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Abstract

The partially ordered set of compact intervals provides a convenient embedding space for the analysis of some Dynamical Systems. Crucial dynamical properties are transferred to it, while allowing an investigation of stability and chaoticity, in terms of computability, in particular in the presence of singularities. We will survey some results which display the connections between the geometric complexity of the dynamics and computability issues, as well as new relations between dynamic predictability and effective decidability.

Keywords: Computability over continuous domains; Decidability; Unpredictability; Chaotic dynamics; Singularities; Shadowing

1. Discrete data types

Computability and modern computers are a direct fallout of the focusing on arithmetic in the foundational analysis of mathematics. Following the explicit project of the founding fathers, Frege and Hilbert, direct relations to physical space had to be excluded from these analyses: Logic is Arithmetic (Arithmetic is Logic), where "everything thinkable may be expressed" [14] and where all geometries may be encoded [17]. "Unshakable certainties" (Hilbert) could thus be reinstated, after the immense crisis of non-Euclidean geometries (a "delirium", as for intuition, said Frege in [14], see also [38,26]).

Along with this arithmetic turn in foundations and in order to understand what could be known (decided), (mathematical) knowledge had to be reduced to the "least act of human thought": write/erase 0 and 1, move right/left. Thus, the Logico-Arithmetic Machine of Turing (1936) was the origin of the extraordinary devices, Discrete State Machines, as Turing later called them (1950); a paradigm for deterministic and theoretically predictable action on discrete data types. Now, how can we deal with physical problems which are traditionally presented by the mathematics of continua, in continuous space and time?

It is well-known that simulating a deterministic chaotic dynamical system (see below for a definition) with a digital computer leads to unreliable results.

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As part of the common knowledge, we know that this is due to:

- the inaccuracy of measurement of the initial conditions (meteorologists are unable to measure the position of every butterfly in the atmosphere)
- the inaccuracy of the computations (meteorologists use machines involving round-off errors).

Although the first problem can hardly be avoided (yet, it may be treated in an accurate way); the second problem can be dealt with and Domain Theory can be a tool for this. This framework enables *exact* computations on *approximations* of data. It thus internalises the finite-precision measurement problem and it allows the deduction of assured results. In short, approximate physical measure is usually understood in terms of *interval arithmetics*, now largely captured by Domain Theory. We will see that the issue is related to the prediction of the "trajectories" in a dynamical system with an approximate initial condition. The theoretical problem is thus the following. As long as the mathematical frame has suitable properties of stability or robustness or even linearity, the machine "follows" the continuous description. Yet, since Poincaré's Three (gravitational) Bodies Problem and, more extensively, since the major results of the '50s and '60s, non-linear dynamical systems have come into the limelight. Usually, non-linearity expresses "interactions", as contrasting agents in a suitable field, or "resonance" effects; that is, the equations of the dynamics express reciprocal actions of the components of the system (three planets interacting in their gravitational fields, ago-antogonistic systems — see below), which permanently "destabilize" the deterministic evolution. This may create chaos and, thus, unpredictability. Turing, in his paper of 1952 on Morphogenesis, [41], beautifully defines the consequence of non-linear interactions on initial data as "exponential drift".

Turing's paper analyzes the genesis of forms in an action/reaction/diffusion system, where minor changes yield radically different evolution. And this is what matters in his analysis: fluctuations below observability generate macroscopically different forms. As he had clearly observed in [40], this property doesn't apply to his Discrete State Machine: the discrete data type gives a least level of observability, the intended discretization (see [27] for a comparative reflection on the two papers). To put it another way, when the intended structure has as a "natural" topology discreteness, the problem of physical measure radically changes: variations or fluctuations below discretization do not participate in the mathematical causal structure of the system. Of course, one may embed a computer's data type by different topologies for good reasons, but the discrete structure of digital data types is at the core of computing.

In summary, two properties pose, since Turing's early remarks, a major challenge, when they are present **conjointly**:

– the exponential drift, a mathematical property of the intended system of determination (later called "sensitivity to initial conditions[1]")
– the physical aspect of "measure", namely approximation.[2]

This conjunction of mathematical and physical principles yields the intrinsic unpredictability, first proved by Poincaré, of deterministic chaotic systems. It is sound to have this unpredictability coincide to randomness, in classical frames, where a good definition of a "random process" may be given as follows: a physical process is random when, iterated in the "same initial conditions" (in the sense of physics), it will not follow the "same" trajectory (in the intended phase space).

We will not pursue this issue here (see [3]), but we will just comment that there is no randomness in (classical as distinguished from quantum) computers: pseudo-random generators are simple iterative programs on the same initial conditions of the discrete data base (here the naturalness of the discrete topology pops out clearly: data are isolated, there is no "natural" neighbor of the approximated physical measure).

[1] See below for a definition. Unfortunately when Ruelle, in the '70s, proposed this very effective – and popular – name for Turing's exponential drift, he was not aware of Turing's morphogenesis paper. That seminal paper was later revitalized in phyllotaxis and biological morphogenesis, where it is considered a founding contribution. "Exponential drift" makes particular sense because in Turing's case and in most interesting ones, some "exponents" crucially appear in the dynamics – the so called Lyapunov exponents – which allow even to measure the degree of "chaoticity".

[2] Note that the fact that measure in – classical/relativistic – physics is always an interval is not a "practical" matter, but a theoretical issue: to say at least, the thermal fluctuation is always present, as perfect rigidity cannot exist both for thermodynamical and relativistic reasons (a perfectly rigid body would instantaneously transmit an impulse).

2. Myhill–Shepherdson Theorem and Scott topologies

Is there continuity in computability theory, an eminently arithmetic theory, as recalled above? The need for continuity is posed in at least two ways. First, there may be good reasons to embed the discrete space in a continuous frame, use the latter to prove theorems and, then, try to project them (or see what they yield) on the discrete fragment. This is an "analytic" approach, if we borrow the name from number theory, which has been largely pursued (see [25]). Second, and not without relation, the countable set of numbers has a continuum of subsets and functions on it. These sets may be naturally endowed with non discrete topologies.

The Myhill–Shepherdson Theorem ([28], see below), in Recursion Theory, is one of the earliest results which provided a "topological", though implicit, characterization of a purely recursion theoretic notion of a functional (or function with function argument). The topological content of this relevant theorem was made explicit and generalized by Dana Scott in the early '70's, for an analysis of typed and type-free computations. Since then, the Scott topologies and their numerous variants have had a major role, both in higher-types of Recursion Theory (where Ershov later rediscovered the same structures, within an original frame, see [23]) and, more importantly, in the Denotational Semantics of programming languages. Along the lines of some recent work by Edalat, we will use these ideas here to look at some dynamics of broad interest for practitioners of dynamical systems. The deep reasons behind this stepping in of (simple) topologies in computability, is that, in mathematics, when working with functions, as sets of input/output values, one has to deal with infinite objects. And the royal way out towards infinity, in constructive systems, is provided by approximation. But approximation means *order* and *topology*, or, at least, a weak form of approximation as "limit" over filter (non-topological) spaces, as suggested by [19] and recently developed by [8] (see also [9]). We refer here to the Domain Theoretic frame, along the lines of Edalat's work, as it seems the most natural for a theoretical approach to Interval Analysis, a largely developed tool in numerical analysis. Moreover, dynamical systems are analyzed as topological or, more often, metric spaces.

2.1. Continuous domains

Recall that computations, since Turing's early results, are essentially partial (the halting problem is undecidable or there exists partially computable functions that cannot be extended to total ones). There is now a natural way to order partial functions as input/output devices: they can be partially ordered by graph inclusion (that is, "smaller" means "less defined"). This is the underlying intuition that motivates the constructions below.

Consider a partial order $D = (D, \leq)$. For the purposes of an analysis of approximation, we need to generalize the notion of "chain" and "limits", in total orders, to partial ones. A subset A of D is **directed** if for x and y in A, there exists z in A such that $x, y \leq z$ (a chain is a particular directed set, of course). The limits will be provided by the suprema of directed sets, $\sup A$, whenever they exist.

Definition 2.1. Let x and y be elements of D, then x is **way below** y (notation: $x \ll y$) if: for any directed set A in D, if $y \leq \sup A$, then $\exists a \in A\ x \leq a$.

Clearly, the relation $\ll$ is finer than $<$, and one has: $x \ll y$ and $x' \leq x$ and $y \leq y'$ imply $x' \ll y'$. Notation: $\Uparrow a = \{b \in D \mid a \ll b\}$; $\Downarrow a = \{b \in D \mid b \ll a\}$.

Definition 2.2. D is a **continuous domain** if, for each $a \in D$, the set $\Downarrow a$ is directed and $a = \sup \Downarrow a$.

In a continuous domain D, the set $\{\emptyset\} \cup \{\Uparrow a \mid a \in D\}$ trivially forms a basis for a topology (which induces the partial order, i.e. $x \leq y$ iff for all open U, $x \in U \Rightarrow y \in U$). Call this topology, the **Scott topology**. A subset D_0 is a **basis**, if, for each $a \in D$, the set $\Downarrow a \cap D_0$ is directed and $a = \sup \Downarrow a \cap D_0$. Clearly, also $\{\emptyset\} \cup \{\Uparrow a \mid a \in D_0\}$ is a basis for the Scott topology. If D has a countable base, then D is **ω-continuous**.

Note that the notion of approximation (or *neighborhood*) is given "from below" and that Scott open sets are upward closed: moving up, along the partial order, gives more and more defined (better approximated) elements [35,36]. These topologies are clearly very "simple" from any mathematical point of view; that is, they are too weak or separate too little. Yet, some robust theorems justify them fully (first, Myhill–Shepherdson classical result). The recent applications mentioned in the rest of this paper give further motivations for their mathematical interest.

Example 2.3. Let $P\mathbb{N}$ be the powerset of $\mathbb{N}$, the set of natural numbers, and P the set of partial (number theoretic) functions. $P\mathbb{N}$ and P are ω-continuous domains by set inclusion, where $a \ll b$ for some b iff a is finite iff $a \ll a$ (finite, in P, means "function with a finite graph", or equivalently "with a finite domain" — of definition; approximation is graph inclusion — thus, the total functions are the maximal elements).

In general, call **algebraic** any element a in a continuous domain, that satisfies the $a \ll a$ relation. This is a general notion of "finiteness" thus extended to functions and functionals in higher types, since ω-continuous domains form a Cartesian Closed Category: that is, it extends the useful properties of "finite elements" in more general settings, where there are no (properly) finite elements. For example, given two continuous domains A, B, the collection $CD[A, B]$ of continuous functions (w.r.t. the Scott topology) may be endowed with the structure of an ω-continuous domain by taking the pointwise convergence topology on it.

However, many interesting domains have no algebraic elements. Here is one that will be used later on.

Example 2.4 (*The* **Upperspace** *of* $[0, 1]$). Consider the collection $\mathbf{I}([0, 1])$ of the compact intervals in the real interval $[0, 1]$. Let $D = (\mathbf{I}([0, 1]), \supseteq)$, that is $\leq$ is the *reverse* inclusion. Then:

- $(\mathbf{I}([0, 1]), \supseteq)$ is an ω-continuous domain (base: the intervals with rational end points).
- $[a, b] \ll [c, d]$ *iff* $[c, d] \subseteq]a, b[$ (note!!)
- the singleton intervals are the largest elements and $[0, 1]$ is the least one.

The intuition should be clear: given a compact interval $\mathcal{I}$ in $\mathbb{R}$, call $\mathbf{I}\mathcal{I}$ the upper space of the compact intervals of $\mathcal{I}$. A basic Scott open is given by the collection $\Box O = \{a \in \mathbf{I}\mathcal{I} | a \subset O\}$, for an open set $O \subset \mathcal{I}$. The least upper bound (lub) of any directed set is the intersection of its elements and the relation of approximation $a \ll b$ is given by $b \subset a^\circ$. No element of this domain is algebraic. The real points are approximated from below by decreasing the size of the intervals containing it. Thus, approximation from below gives the reals as fully specified elements (the total ones, in a sense); that is, a real number x is approximated by the shrinking sequence of rational nested intervals. This is fully coherent with Interval Analysis, as the induced topology from $\mathbf{I}\mathcal{I}$ over $\mathcal{I}$, the set of maximal elements, is the real topology and the mapping $s : \mathcal{I} \to \mathbf{I}\mathcal{I}, x \mapsto \{x\}$ is an embedding of $\mathcal{I}$ onto the set of maximal elements as $s^{-1}(\Box O) = O$ for any open subset $O \subset \mathcal{I}$. A continuous function $f : \mathcal{I} \to \mathcal{I}$ can be extended to a Scott continuous function $\mathbf{I}f : \mathbf{I}\mathcal{I} \to \mathbf{I}\mathcal{I}$ as $(\mathbf{I}f)(a) = f(a)$ for a compact interval a.

A countable basis is given by the set of all intervals with rational endpoints and the natural enumeration of rational numbers gives an effective structure to $\mathbf{I}\mathcal{I}$. A computable real number is then the lub in $\mathbf{I}\mathcal{I}$ of a sequence of rational intervals generated by a program. A computable function is the lub of a computable sequence of rational functions of the domain of Scott continuous functions $\mathbf{I}\mathcal{I} \to \mathbf{I}\mathcal{I} = CD[\mathbf{I}\mathcal{I}, \mathbf{I}\mathcal{I}]$. Over the reals, these functions coincide with those induced by the Enumeration Operators in [32]. More generally, these definitions correspond to the computable real functions of Recursive Analysis [31]; this may be shown by a notion of type-2 Turing Machine as given in [42] (a recent survey to computability on the reals may be found in [4]). Our approach focuses on (real) intervals and the associated domain, in view of their role in representing approximation in physical dynamics, which is the aim of this paper (sections 3–5).

2.2. Generalized Rice–Shapiro Theorem

Consider now an ω-continuous domain D. Then its basis D_0 can be enumerated by a bijection $e : \mathbb{N} \to D_0$, say. The domain is called **effective** if $e(n) \leq e(m)$ and whether there exists a p such that $e(n), e(m) \leq e(p)$ are decidable predicates in n and m. This is a key property as for effectiveness, as it clearly applies to our leading examples above, in particular $P\mathbb{N}$ and P, and it allows us to define the collection of computable elements in an effective domain: just take the limit (the sup) of all effectively enumerable directed sets in D_0. That is, consider the directed subsets A which possess a recursively enumerable (r.e.) set of indices, i.e. $\{n \,|\, e(n) \in A\}$ is r.e.. Their suprema, if they exist, will be the generalized computable functions and functionals in (higher) types or in abstract structures. In $P\mathbb{N}$ and P, these sets give exactly the r.e. sets ($RE \subseteq P\mathbb{N}$) and the partial recursive functions ($PR \subseteq P$). In $(\mathbf{I}([0, 1]), \supseteq)$ they are the intervals with a computable from below real number, as a left end point, and a computable from above real, as a right end point (a real is computable from below (resp. from above), when it is the limit of a nondecreasing (resp. nonincreasing) effective sequence). Thus, if one starts by considering TM's and their computable functions (PR or

RE), we are now seeing them as embedded into suitable continua (P or $P\mathbb{N}$), endowed with an order topology (note that both $(P, \subseteq)$ and $(P\mathbb{N}, \subseteq)$ contain chains with the order structure of the reals).

Is this construction of a category of domains, really "natural"? At least a theorem with two important corollaries proves it, by establishing a nice correspondence between topologies and computations. In order to state them, observe first that it is easy to enumerate the limits or sups of r.e. directed sets of the base, if they exist: just use the indices of the r.e. set, which enumerate the directed set below it. Each of these limits, of course, will have infinitely many indices, at least as many as the indices of enumerating r.e. set. In an effective ω-continuous domain D, call D_c the set of the computable elements in D, and $w : \mathbb{N} \to D_c$ this (canonical) enumeration of D_c. An effective domain D is called **effectively complete** if all limits of the r.e. directed sets of the base exist in D_c. Then one has:

Theorem 2.5. *In an effectively complete ω-continuous domain D, consider a canonical enumeration $w : \mathbb{N} \to D_c$ and $B \subseteq D_c$. Then, if $\{n \mid w(n) \in B\}$ is recursively enumerable, one has that B is an open set (in the induced Scott topology on D_c). Conversely, for any trace $B \subseteq D_c$ of an open set of the basis, the set $\{n \mid w(n) \in B\}$ is recursively enumerable.*

The first implication is a non-obvious fact, with several consequences (the converse is easy). Observe first that there is no apparent reason by which a recursion theoretic notion, the recursive enumerability of a set of indices, should lead to topological openness. Look at this theorem in the special cases of $P\mathbb{N}$ or P, where it was first proved by Rice and Shapiro (see Theorem 14.XIV in [32]) by a non-trivial argument: its key lemma, later called "effective discontinuity", has a relevant logical-intuitionistic meaning, and it may be also reconstructed from the proof of Myhill–Shepherdson Theorem (the generalization is easy).

A map between domains $F : D_c \to D'_c$ is **computable**, with respect to enumerations $w : \mathbb{N} \to D_c$ and $w' : \mathbb{N} \to D'_c$, if there exists a recursive function f such that $F(w(n) = w'(f(n))$ (that is, F is computed over the indexes by f). As a diagram:

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{f} & \mathbb{N} \\ w \downarrow & & \downarrow w' \\ D_c & \xrightarrow{F} & D'_c \end{array}$$

The (Generalized) Myhill–Shepherdson Theorem, analyzes, in terms of continuity, computable functionals of function input (and in all higher types). We can prove it now as an immediate corollary of the theorem above, as the hard part consists in proving the *continuity of computable functionals* (why should they be continuous?). This easily follows now from the following facts:

– (basic) open sets are (enumerated by) r.e.,
– computable functions (over the indexes, such as f say) backward transform r.e. sets in r.e. sets
– r.e. sets are (enumerate) open sets.

The theorem above implies also that if you take a collection of r.e. sets or of partial recursive functions, then their semi-decidability gives a certain order or topological structure to the set of r. e. sets or functions, considered in extenso (as set-theoretic relations, e.g. as elements of $P\mathbb{N}$ or P): they must be topologically open. As an immediate further consequence, one has that a property of r.e. sets or partial recursive functions is decidable iff it is trivial (just use the recursive enumerability of a recursive set and of its complement and the existence of a least element in a domain). More formally:

Corollary 2.6. *In an effectively complete ω-continuous domain D, with $w : \mathbb{N} \to D_c$ and $B \subseteq D_c$, $\{n \mid w(n) \in B\}$ is recursive iff B is empty or coincides with D.*

The relevance of the Corollary should be clear, even in the basic case of $P\mathbb{N}$ or P, where the result was originally given in [33]. In summary, no non-trivial property of programs is decidable (Corollary 2.6), and the semi-decidable properties are "very few" or "topologically very structured" (Theorem 2.5). In particular, we cannot decide nor semi-decide whether a program is correct, that is if it computes a given function or a function in the desired set. This intrinsic limitation started the enormous amount of work on program correctness in Computer Science.

It should be clear that we are not following history here. As a matter of fact, the Rice theorem was proved first, as a consequence of the halting problem [33], then Myhill–Shepherdson and Rice–Shapiro theorems followed, but just over $P\mathbb{N}$ and P, with no explicit reference to topology (see [32]).

2.3. *Computability on the reals*

Let $\mathcal{I}$ be a compact interval in $\mathbb{R}$. The effective structure of the upperspace $\mathbf{I}\mathcal{I}$ is given by a enumeration $e : \mathbb{N} \to \mathbf{I}\mathcal{I}$ of the compact rational intervals, extended to a representation $w : \mathbb{N} \to \mathbf{I}\mathcal{I}$ of the computable elements of $\mathbf{I}\mathcal{I}$ (compact intervals, including singletons).

Recall now that a function $f : \mathbf{I}\mathcal{I} \to \mathbf{I}\mathcal{I}$ is **computable** if there is a Turing machine φ such that the following diagram commutes :

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\varphi} & \mathbb{N} \\ w \downarrow & & \downarrow w \\ \mathbf{I}\mathcal{I} & \xrightarrow{f} & \mathbf{I}\mathcal{I} \end{array}$$

A function $f : \mathbb{R} \to \mathbb{R}$ is computable if its canonical extension on $\mathbf{I}\mathcal{I}$ is computable.

A function $f : \mathbf{I}\mathcal{I} \to \mathbf{I}\mathcal{I}$ is *strongly* computable if it is computable and if the domain of $f \circ w$ (the set of integers on which it is defined) and the domain of φ (the set of integers on which the machine halts) coincide.

A set $A \subset \mathbb{R}$ is *semi-decidable* if there is a strongly computable function $f : \mathbf{I}\mathcal{I} \to \mathbf{I}\mathcal{I}$ which domain's intersection with $\mathbb{R}$ is A ($\mathbb{R}$ being assimilated with the set of singletons of $\mathbf{I}\mathcal{I}$). A property of the reals is semi-decidable if its characteristic set is semi-decidable.

As a consequence of the Generalized Rice–Shapiro theorem, recall that every semi-decidable set is open for the Euclidean topology (which is the "trace" on $\mathbb{R}$ of the Scott topology of $\mathbf{I}\mathcal{I}$).

3. Dynamical systems

We are interested in systems with discrete time steps; our purpose is to understand the asymptotic behavior of an iterative process: x, $f(x)$, $f^2(x)$, $f^n(x)$, ... as n goes to infinity. Here f is a function on a closed interval $\mathcal{I}$ (more generally: on a compact metric space). This sequence $\langle f^n(x)\rangle_n \in \mathbb{N}$ is called the **orbit** of a point $x \in \mathcal{I}$. A point x is called a **fixed point** if $f(x) = x$. It is a **periodic point** of period n if there exists $n \in \mathbb{N}$ such that $f^n(x) = x$, the smallest such n is called the principal period of x. The orbit of such a point is called a periodic orbit.

Given a dynamical system $(f, \mathcal{I})$, a compact set A is an **attractor** of the function f if there exists an open neighborhood V of A and $k \in \mathbb{N}$ such that $f^k(V) \subset V$ and $A = \bigcap_{n\in\mathbb{N}} f^n(V)$. A set A is an **invariant** for the function f if $f(A) \subset A$.

The main purpose of studying dynamical systems is to understand the nature of the orbits. The computation of orbits on a computer is not very efficient especially when we consider the accumulation of round-off errors. So we are more interested in qualitative properties such as the structure of the attractor and stability. A periodic point p of period n is **stable** if there exists an open interval U containing p such that $lim_{i\to\infty} f^{ni}(x) = p$ for all $x \in U$.

A function $f : J \to J$ is said to be **topologically transitive** (or transitive) if for every pair of open subsets $U, V \subset J$, there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$. A function $f : J \to J$ has **sensitive dependence on initial conditions** (or is sensitive) if there exist $K \subset J$ with $\mu(K) > 0$ where μ is a Lebesgue measure, and $\delta > 0$ such that, for all $x \in K$ and for all neighborhood V of x, there exist $y \in V$ and $n \in \mathbb{N}$ such that $|f^n(x) - f^n(y)| > \delta$.

Intuitively, transitivity means that the iterates of an open set will eventually intersect any other open set (thus, there exist dense orbits, an important witness of chaos). Sensitivity means that any two points, no matter how close, will eventually be apart by δ under successive iterations (as second element of chaos, which contributes to unpredictability: points, so close to be unobservably apart, may radically diverge ... possibly by an exponential drift). As we said, from a practical point of view, those functions pose many problems in their simulation on computers due to round-off errors.

A dynamical system is **chaotic** when it is topologically transitive, sensitive to initial conditions and has a dense set of periodic orbits. In other words, it is unpredictable, indecomposable but still regular (in some way: the density of periodic orbits is a form of regularity).

3.1. Dynamics and pointwise computability

Dynamical systems and their extensions to domains have common properties, chaoticity in particular. Consider the poset $\mathbf{I}\mathcal{I}$ of the compact intervals of $\mathcal{I}$ as an ω-continuous domain. On the compact interval $\mathcal{I}$ a continous map $f : \mathcal{I} \to \mathcal{I}$ is chaotic if and only if it has a continuous chaotic extension $g : \mathbf{I}\mathcal{I} \to \mathbf{I}\mathcal{I}$, with respect to Scott topology, [12]. We first focus on the purely mathematical part of a dynamical system, detached from the physical modeling which gave birth to it. Concretely it means that we work with exact initial conditions and investigate checkability of some properties of their orbits. Domain Theory will still be useful because it enables *exact* computation on *exact* data. As we have seen, a (computable) real number, as an exact data, may be presented as the (effective) limit of a sequence of rational compact intervals (approximations). Thus, computations on the rationals naturally extends to computations on the reals and allow us to define, as we did, computable functions on reals.

More specifically, given a computable function $f : \mathbb{R} \to \mathbb{R}$ and a computable initial condition $x_0 \in \mathbb{R}$ represented by an index i_0 ($w(i_0) = x_0$), the orbit of x_0 under f can be computed exactly, i.e. the indexes of the iterates can be computed. This implies that the iterates can be computed to any step and with arbitrary precision: there is no round-off, as computable reals are represented by the index of one of their generating functions. Thus the problem of exponential drift associated with chaos is solved by this computational model. The problem left is the computation time to approximate the iterates. Actually, chaos makes the computational complexity explode exponentially with the number of iterates and with the precision. Yet, this approach allows us to make a preliminary decidability remark.

Given a dynamical system and an initial point for a trajectory, one may ask, and try to decide, whether that point is taken into a given neighborhood.

Theorem 3.1. *Given any computable function $f : I \to I$ (with $I \subseteq \mathbb{R}$) and an open interval V with rational endpoints, the set of points reaching V:*

$$\{x \in I \,|\, \exists n, f^n(x) \in V\}$$

is semi-decidable.

This result is easy and general: it holds for any computable function. In practice, the more chaotic the function is, the longer it takes to semi-decide this set. We will examine below more physically interesting and computationally feasible questions.

3.2. Robustness and Chaos vs semi-decidability

As we have seen, chaoticity of dynamics a priori involves (major!) complexity problems as for computations. But we will see now that it also causes limitations in the semi-decidability of some properties, particularly a (non-)reaching property (does an orbit get into – does it reach – a given interval?). To do so, we compare what happens for a chaotic system and for a robust system. This brings us to the more realistic handling of computations by round-off.

The idea underlying the definition below is the following: a system is robust if a simulation with a round-off computer is "reliable". Now, if δ is the round-off error of the computer, a sequence computed by the machine is not a "real" orbit, but a δ-pseudo-orbit[3]:

Definition 3.2. Given a dynamical system (I, f), with $f : I \to I$, and a $\delta > 0$, a *δ-pseudo-orbit* for f is a sequence $(x_i)_{i \geq 0}$ such that for every i:

$$|x_{i+1} - f(x_i)| \leq \delta.$$

We now define robustness, which is a very strong requirement, because it implies that the dynamics "resists" perturbations.

Definition 3.3. A dynamical system (I, f), with $f : I \to I$ is *robust* if for every $\epsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit $(x_i)_{i \geq 0}$ is ϵ-close to the real orbit starting from x_0, i.e.:

$$|x_i - f^i(x_0)| \leq \epsilon \quad \text{for all } i.$$

[3] Clearly, by a real orbit, we mean a continuous space, discrete time trajectory; while a pseudo-orbit is meant to be, in this computational approach, a discrete space, discrete time trajectory)

A sufficiently precise computer (with the "good" δ) can then predict a robust system up to a given precision ϵ. In particular, one may semi-decide whether an orbit will "keep away" from a given neighborhood.

Definition 3.4. Given a computable dynamical system (I, f), the *keep away* relation is defined as:

$$\begin{aligned} KA(x, V): &\iff \text{the orbit of } x \text{ keeps away from } V \\ &\iff \inf_{n\in\mathbb{N}} d(f^n(x), V) > 0 \end{aligned}$$

where x ranges over I and V over rational compact intervals.

Robustness or chaoticity of the dynamics of a system lead to a key difference in computability:

Theorem 3.5. *Given a computable dynamical system* (I, f)*:*

- *if f is robust, KA is semi-decidable,*
- *if f is chaotic, KA is not semi-decidable.*

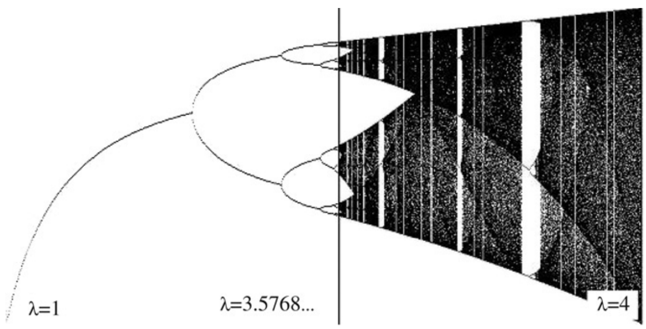
Remark. The rational interval V can be viewed as a parameter encoded into an integer, written on an additional tape of the Turing machine semi-deciding KA.

Proof. Hint: use the relation between topological openness and recursive enumerability obtained by using domains. □

More on this may be found in [5,6], details on reachable sets in [7] (see [18] for the proofs of the theorems above).

4. The logistic maps

A well known example of chaos, is provided by the quadratic function on the unit interval $\mathcal{I}$: $f_4(x) = 4x(1 - x)$ (see [10] or most books on Dynamical Systems). This chaotic function is a member of the family of logistic functions : $f_\lambda(x) = \lambda x(1-x)$. It is a discrete-time (and continuous-space) representation of the famous Lotka-Volterra equation for ago-antagonistic processes (discrete-time, because one looks at the iterates f^n on an initial real point, as usual). We will focus on the logistic function in this section, the main one of the paper, as a paradigmatic case of non-linear dynamics. Both the classical equation and the time-discrete version have been largely applied in physics and biology (a growth x non-linearly opposes ecological constraints, $(1 - x)$). The coefficient λ has a major relevance: different values of λ yield very different behavior (for more on the dynamics of these functions see [20]): we shall see that. These maps are generally considered paradigmatic dynamics, also because many problems in several dimensions may be reduced to one-dimensional ones: in the analysis of two or more planets, say, the only thing that matters for their viable "stability" is the distance from the Sun.



This image is computer-generated and it is the trace of 1000–2000 iterates of the logistic maps, with ranging coefficients λ, between 1 and 4.

The logistic function f_λ has a simple behavior with a stable fixed point for $\lambda \in [0, 3)$ and all interior points of $\mathcal{I}$ converge to that point. It has a chaotic behavior for $\lambda = 4$. In order to understand what happens between these two extremes, let's recall a general idea. In a dynamical system, a bifurcation occurs when the behavior of a parametrized function undergoes a qualitative change as a result of a change in the parameter. More precisely, a **period doubling**

bifurcation occurs when a stable fixed point becomes unstable and a stable periodic orbit is created about it. In the picture above, for $\lambda = \lambda_1 = 3$, the fixed point becomes unstable and a stable periodic orbit is formed. That is, for $\lambda = 3$, a period doubling bifurcation takes place and a stable periodic orbit of period 2 is generated, while the fixed point remains, but it loses stability. Then, for $\lambda = \lambda_2 \approx 3.5$, this orbit becomes also unstable and a new stable periodic orbit of period 4 is generated. For increasing values of λ the bifurcation iterates and produces stable orbits of period 4, 8, 16 .... At each bifurcation, the previous periodic points become unstable (and, thus, are invisible at the computation when erasing the first 1000 iterations: the machine doesn't show unstable fixed or periodic points, when iterating, because of the round-off).

This scheme repeats infinitely many times for all λ_n. We define

$$\lambda_\infty = \lim_{n\to\infty} \lambda_n = 3.5768\ldots.$$

Remark 4.1. [For the reader with some knowledge on "Renormalization Techniques" in critical transitions, see [18] for details] $\lambda = \lambda_\infty = 3.5768\ldots$ happens to be a computable real. We hint here an argument for its computability.

Just after the first period-doubling (at $\lambda = 3$), f_λ^2 has two stable fixed points (which correspond to a period 2 orbit). Each of these fixed points is an extreme of an invariant interval, on which f_λ^2 has the same shape as a $f_{\lambda'}$ with λ' larger than 1, but close to 1. We then say that f_λ^2 can be "renormalized": in short, renormalization consists in restricting our view of f_λ^2 on some invariant interval. As this renormalized map has the same shape $f_{\lambda'}$, the study of the latter gives complete information on the former.

So, when λ grows from $\lambda_1 = 3$, the renormalization of f_λ^2 (denoted Rf_λ) behaves like $f_{\lambda'}$ with λ' growing from 1. Thus, for some λ, the critical point of Rf_λ is mapped onto the maximal end point of its interval of definition. Then f_λ^2 is no longer renormalizable, because the restriction interval is no longer invariant. This renormalization failure occurs at some $\lambda_2 < 4$ (for which Rf_λ looks like f_4).

To sum up, for some λ_1, λ_2 with $1 < \lambda_1 < \lambda_2 < 4$, f_λ^2 is renormalizable iff $\lambda \in [\lambda_1, \lambda_2]$.

We can iterate this argument: there are λ_3, λ_4 with $\lambda_1 < \lambda_3 < \lambda_4 < \lambda_2$, such that f_λ^4 is renormalizable if $\lambda \in [\lambda_3, \lambda_4]$. And so on ....

We obtain two sequences $< \lambda_{2n+1} >, < \lambda_{2n+2} >$ converging respectively from below and from above to λ_∞. These sequences are computable (their elements are solutions of effectively produced algebraic equations). Then λ_∞ is a computable real number. It appears not to be known if, as a computable real, it is irrational or transcendental — and the question is not computationally irrelevant, as checking the equality of two algebraic equations may be decided or not according to the transcendence or not of λ_∞.

Let's go back to the geometric and dynamic properties of λ_∞. At $\lambda = \lambda_\infty$, the function $f = f_\lambda$ (called **Feigenbaum function**) has a quasi-chaotic behavior: it possesses only *unstable* periodic orbits of period 2^n for all $n \in \mathbb{N}$. Moreover, it is topologically transitive (a component of chaos), but it is not sensitive to initial conditions. It has an attractor of the form $A = \bigcap_{n\in\mathbb{N}} I^n$. Here I^n is given by $I^n = \bigcup_{1\le j\le 2^n} I_j^n$, with $I_j^n \in \mathbb{I}\mathcal{I}$ is the interval with endpoints $f^j(1/2)$ and $f^{j+2^n}(1/2)$. We have $I^{n+1} \subset I^n$ and A, the set of the maximal elements of the domain, is a Cantor set.

In this list of non-obvious properties of f_λ at $\lambda = \lambda_\infty$, we observe that every non-periodic point $x \in]0, 1[$ will eventually be in every I^n. Moreover, a remarkable "phase transition" takes place at λ_∞: from countable to continuous infinite. As a matter of fact, we reach λ_∞, from the left, by a cascade of finite bifurcations and obtain a countably infinite limit. From the right, but we have no space to show this here, we reach λ_∞ by a Cantor-like fractal structure, thus of measure 0, but of uncountable cardinality. By this, the transition goes, on one point, from countable to uncountable infinity.

We have seen that the function has a stable periodic orbit for each $\lambda < \lambda_\infty$. As for its behavior at parameter values greater than λ_∞, the next theorem shows that this holds for most values in $[0, 4]$. That is, surprisingly enough, even to the right of λ_∞, one may find "arbors of peace" (stable periodic orbits), almost everywhere.

Theorem 4.2. *The set* $S = \{\lambda \in [0, 4] \mid f_\lambda$ *has a stable periodic orbit*$\}$ *is locally connected and dense in* $[0, 4]$

The very complex morphology of the dynamics generated beyond λ_∞ is also given by this further result:

Theorem 4.3. *For all* ε *there exist* $n \in \mathbb{N}$ *and* λ_a, λ_b, *with* $4 - \varepsilon < \lambda_a < \lambda_b < 4$, *such that* f_{λ_a} *has a stable periodic orbit of period* n *and* f_{λ_b} *is a chaotic function. Moreover, as the parameter* λ' *increases from* λ_a *to* λ_b, *the function*

$f_{\lambda'}$, restricted to a finite union of intervals $I_{\lambda'} \subset \mathcal{I}$, follows, on each interval, exactly the same pattern as f_λ when λ increases from 2 *to* 4.

This non-obvious theorem proves that arbitrarily close to 4, one can find copies of what happens in the interval [2, 4] (!). For a proof of both results, see [21].

4.1. Semi-decidability

From the discussion (and the image) above, it should be clear that the dynamical behavior of f_λ changes dramatically according to the value of λ. The results above show the alternating richness and the geometric complexity of these changes. The point is that, different dynamical and geometric features lead to different computability effects:

Theorem 4.4. *Let's consider the logistic map f_λ, with λ a computable real number (so that f_λ is a computable function). We have:*

- *for $\lambda < \lambda_\infty$, the set of non-periodic computable points of f_λ is semi-decidable, but not decidable,*
- *for $\lambda \geq \lambda_\infty$, in chaotic and Feigenbaum cases, the set of non-periodic computable points of f_λ is not semi-decidable, nor its complement.*

Thus (see also the next section), the dynamical complexity of a system has a direct consequence on the semi-decidability of these key properties. Through computational complexity effects, chaos makes practical simulations (also with Domain Theory) very hard, but it also prevents some properties from being (semi-)decidable, as in the theorem above (see [18] for the proofs, which use Domain Theory, as a frame for generalizing 2.5).

4.2. Predictability

We observed that "predicting" is a difficult issue, in dynamical systems, in view of the joint problem of sensitivity and of approximation in physical measure. Now, the prediction problem for a physical system consists in answering questions on a possible trajectory, while knowing the initial position approximately. More precisely, we cannot know the initial point as a real number, but as an interval, and we do not want to know the entire trajectory, but we can ask whether a trajectory, starting within a measure interval, will pass through a given *zone*.

The approximation by the physical measure may then be expressed, mathematically, by giving a neighborhood U which represents the best measure we can have in the intended physical system. An interesting problem then may be to try to predict whether this neighborhood is, sooner or later, taken into a target one, V, say.

For this purpose, we state our main problem as follows: Given a dynamical system as a function f acting on a topological space I and two subsets (neighborhoods) U and V, is there a time t such that $f^t(U) \subset V$?

The challenge, of course, is due to the fact, related to chaoticity, that non-linear maps are "mixing". On a real interval $\mathcal{I}$, this means that extremes are not transformed into extremes (of the initial interval) and maxima and minima are not preserved.

In the case of the logistic family of functions, the problem is: given a function $f_\lambda : \mathcal{I} \to \mathcal{I}$ and two neighborhoods, the initial and final ones, do the iterates of the initial neighborhood under f_λ ever get into the final neighborhood?

We split the question of predictability in these three types of possible behavior of the logistic function f_λ, depending on λ (they are the only possible ones, [20]):

(1) f_λ has a stable periodic orbit.
(2) f_λ has a chaotic behavior (has sensitivity to initial conditions).
(3) f_λ has no stable periodic orbits nor sensitivity (Feigenbaum case).

If we look at this problem in domain theoretical terms, we have a dynamical system on the domain of intervals $\mathbf{I}\mathcal{I}$. Our problem can then be stated as: Given a function $g : \mathbf{I}\mathcal{I} \to \mathbf{I}\mathcal{I}$ and $u, v \in \mathbf{I}\mathcal{I}$ is there an $n \in \mathbb{N}$ such that $g^n(u) \leq v$? In view of the previous remarks, we will take g to be the extension $\mathbf{I}f$ of a given $f = f_\lambda$ on I.

If we think of this problem in terms of computability (as we try to simulate it on a computer) we encounter several well known problems. A major one is to determine the behavior of the function, namely its sensitivity. By Theorem 4.2 we can see that on the interval of parameters, the sets corresponding to different behaviors of the functions have a very complex structure. It can be very hard to determine effectively the behavior of the function from its parameter. Thus, our analysis will be like an oracle giving us which of 1, 2 or 3 above holds.

For the values of the parameter below λ_∞ the structure of unstable periodic points is simple. However, in view of Sharkovsky's theorem [10], this structure may get very complicated for greater values (this a major problem in mathematical analyses of dynamical systems, see [22]). When looking at the dynamics of intervals, the unstable periodic points have an important effect on the behavior of the orbits of the intervals. For instance, in case of a stable function most of the *points* in the interval $\mathcal{I}$ will be converging to a stable orbit, whereas most of the *intervals* in the corresponding domain $\mathbf{I}\mathcal{I}$, are divergent. So, we will restrict our analysis on dynamics of intervals to a subdomain B_λ of $\mathbf{I}\mathcal{I}$ on which the function will have a more regular behavior. It is possible to find such a subdomain in any of the three cases mentioned above, see [21]. But, as we will hint below, the definition and properties of this B_λ is different in each case.

We are thus interested in the decidability of the question corresponding to the predictability problem above. A detailed proof of these results, whose case analysis is only hinted below (following the cases (1)–(3) above, depending on λ), can be found in [21]. (The proof uses Domain Theory as a language only. However, as one may see also from [12], the insight provided by an analysis by intervals allows us to "see" the dynamics and better – just heuristically better – describe the very complex structure of the critical case, in particular).

Problem $\mathscr{P}$. Given a logistic function f_λ, $u \in B_\lambda$ and $v \in \mathbf{I}\mathcal{I}$, both with rational endpoints, is there an $n \in \mathbb{N}$ such that $\mathbf{I} f_\lambda^n(u) \le v$?

(1) *Stable case.*

In this case f_λ has a stable periodic orbit on $\mathcal{I}$. By Singer's theorem [16], there exists an interval $A = [1/2, p]$, where p is a stable periodic point, such that for all $x \in A$ $\lim_{k\to\infty} f_\lambda^{kn}(x) = p$. Define now $B_\lambda = \bigcup_{i=0}^{n-1} f_\lambda^i(A)$. B_λ is an invariant of f_λ in the basin of attraction of the set of stable points.

Given the stability of the function (by an oracle), we can then use Newton's method to effectively determine the set of stable periodic points, and therefore the set B_λ.

If λ is a rational number, by computing the iterates of u, we can decide Problem $\mathscr{P}$. If λ is a computable number, $\mathscr{P}$ is only semi-decidable.

(2) *Chaotic case*

When $\lambda = 4$ (the simplest case), the function has a chaotic behavior on I [10]. Thus, we take $B_4 = \mathcal{I}$.

As the function is chaotic on the whole interval $\mathcal{I}$, for any $u \in \mathbf{I}\mathcal{I}$ there will be an $n \in \mathbb{N}$ such that $\mathbf{I} f_\lambda^n(u) = \mathcal{I}$. As u has rational endpoints, by computing exactly its itinerary we can determine this n. So we will only need to verify for natural numbers $i < n$ if $\mathbf{I} f_4^i(u) \le v$. We can thus conclude that Problem $\mathscr{P}$ is decidable.

More generally, we consider Problem $\mathscr{P}$ for an arbitrary λ where f_λ has chaotic behavior on a subdomain B_λ (by the theorems above, there are many such values of λ). If λ is a rational number then we can compute exactly the iterates of u. As any $u \in B_\lambda$ is eventually periodic under $\mathbf{I} f$, we only need to check if any of those finitely many intervals is a subinterval ov v. Therefore, Problem $\mathscr{P}$ is decidable.

If λ is an arbitrary computable real, by proceeding in the same way we can conclude that Problem $\mathscr{P}$ is again only semi-decidable.

(3) *Feigenbaum case*

In the "main" case, $\lambda = \lambda_\infty$, we can take $B_\lambda = \mathcal{I}$. In this case, u will also be an eventually periodic point of $\mathbf{I} f_\lambda$. And the problem of prediction is reduced to a finite number of cases as in the chaotic case. In particular, if λ_∞ is a rational number (an open problem, apparently) then Problem $\mathscr{P}$ is decidable and otherwise semi-decidable. In a similar way, we can obtain this result for other parameters λ where f_λ is a general Feigenbaum function.

5. Conclusion and further work

The computational analysis carried on here is just one possible direction of an increasing research effort both in Dynamical Systems and in Computable Analysis.

We tried to look at these dynamics in terms of decidability and predictability, in reference to some physical meaning. Yet, too little is known, from the theoretical point of view, about computability aspects of the mathematical modeling of continuous dynamics, in particular concerning transition and emergence. In Numerical Analysis, some results guarantee the plausibility of the simulation: one should quote here more results about robustness and various forms of "Shadowing Lemmas" (see [30]). In short, in some systems (the "hyperbolic ones"), one has:

Theorem 5.1. *Given an hyperbolic dynamical system (I, f), with $f : I \to I$, for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every δ-pseudo-orbit there is a real orbit ϵ-close to it (i.e. ϵ-close at each iteration).*

The meaning of this relevant result should be clear: it doesn't say that in the intended systems the continuous orbits can be approximated (shadowed) by discrete ones, as this is generally false in non-linear systems. It "only" says that, for each discrete (possibly computational pseudo-)orbit, there exists a (space) continuous one that approximates (shadows) the given discrete one (this reversed order is rather important!); moreover, many interesting systems do not realize hyperbolicity nor the theorem, see [34]. Many questions may be raised here about the effectiveness of these constructions as well as about a theoretical treatment of the round-off in terms of continuous domains.

We observe though that our interest is largely motivated by the major difficulties that computations encounter not only in chaotic cases but also in transitions to chaos. In particular, the later case, which is beautifully represented by the Feigenbaum values, such as λ_∞, for the logistic function, presents some peculiar and very general features.

The transition by λ_∞ has many of the mathematical characteristics of a *phase transition* in physics: it is thus "critical" in a very complex sense. Along with changes in a control parameter, λ, an observable or several ones undergo major variations. Here, a cardinality is dramatically modified and a regular behavior (periodic orbits) is transformed in an "almost" chaotic one. Thus, minor fluctuations of the control parameter around λ_∞ induce huge modifications. Yet, some new order emerges: the fractal structure of the Cantor like set, the attractor we examined above, is a form of complex order (some sort of new "coherence structure", as the physicists call it). And this order is reached by symmetry breaking all along the path towards λ_∞, for increasing λ: at each bifurcation, a stable orbit (with its own symmetries) is destabilized and a new symmetry emerges, under the form of a stable periodic orbit with a larger period. As the previous periodic orbit still remains, but becomes unstable, minor fluctuations of x, beyond the bifurcation point for λ, induce a breaking/reconstruction of symmetries.

Now, symmetry breaking in non-equilibrium dynamics is a fundamental procedure in understanding the emergence of a new order and of structures with increasing complexity. And the general paradigm is exactly the one described by this unidimensional case: the emergence of order by symmetry breaking is understood in terms of phase transitions reached by bifurcating trees of local equilibria.

We have no space to discuss these issues here, but the entire area deserves further attention, both because of the increasing role of computational models in physics and because of the general interest of these approaches to "emergence" in the natural sciences, where an "extended notion" of criticality may play some role [2].

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(Preliminary or revised versions of Longo's papers are downloadable from: http://www.di.ens.fr/users/longo or Search: Google: Giuseppe Longo).

References

[1] F. Bailly, G. Longo, Mathématiques et sciences de la nature. La singularité physique du vivant, Hermann, Paris, 2006. English introduction, downloadable.
[2] F. Bailly, G. Longo, Extended critical situations: The physical singularity of life phenomena, Journal of Biological Systems (2008) (in press).
[3] F. Bailly, G. Longo, Randomness and determination in the interplay between the continuum and the discrete, Mathematical Structures in Computer Science 17 (2) (2007) 289–307 (special issue). (largely revised version of the appendix to the book: [1]).
[4] O. Bournez, Modèles continus. Calcul. Habilitation à diriger les recherches, LORIA, Univ. de Nancy, Décembre, 2006.
[5] V. Brattka, G. Presser, Computability on subsets of metric spaces, Theoretical Computer Science 305 (2003) 43–76.
[6] V. Brattka, K. Weihrauch, Computability on subsets of Euclidean space I: Closed and compact subsets, Theoretical Computer Science 219 (1999) 65–93.
[7] P. Collins, Continuity and computability of reachable sets, Theoretical Computer Science 341 (2005).
[8] F. De Jaeger, Calculabilité sur les réels. Thèse en Informatique (dir.: M. Escardo, G. Longo), Université de Paris VII, Novembre 2002.
[9] F. De Jaeger, Computability of the functionals over the reals with the effective filter spaces, Topology Proceedings 26 (2) (2001–2002).
[10] R.L. Devaney, An introduction to Chaotic Dynamical Systems, Addison-Wesley, 1989.
[11] A. Edalat, Dynamical systems, measures and fractals via domain theory, Information and Computation 120 (1) (1995) 32–48.
[12] A. Edalat, Domains for computation in mathematics, physics and exact real arithmetic, Bulletin of Symbolic Logic 3 (4) (1997) 401–452.
[13] Yu.L. Ershov, Model C of partial continuous functionals, in: R. Gandy, M. Hyland (Eds.), in: Logic Colloquium, vol. 76, North-Holland, Amsterdam, 1976.

[14] G. Frege, The Foundations of Arithmetic, 1884. Engl. transl. Evanston, 1980.
[15] G. Gierz, et al., A compendium of Continuous lattices, Springer-Verlag, Berlin, 1980.
[16] J. Guckenheimer, Sensitive dependence to initial conditions for one dimensional maps, Comm. Math. Phys. 70 (1979) 133–160.
[17] D. Hilbert, Les fondements de la géométrie, 1899 (trad. fran., Dunod, 1971).
[18] M. Hoyrup, Dynamical systems: Stability and simulability, in: 3-body, classical-quantum, discrete-continuum, Mathematical Structures in Computer Science 17 (2) (2007) 247–260 (special issue).
[19] M. Hyland, Filter spaces and continuous functionals, Annals of Mathematics and Logic 16 (1979) 101–143.
[20] A. Katok, Introduction to the Modern Theory of Dynamical Systems, Cambridge, 1995.
[21] A. Kolcak, Physical predictability and the dynamics of intervals, Information and Computation, 2008 (Draft, in preparation).
[22] J. Laskar, La stabilité de système solaire, Chaos et déterminisme, Seuil, 1992.
[23] G. Longo, E. Moggi, The hereditary partial recursive functionals and recursion theory in higher types, Journal of Symbolic Logic 49 (4) (1984) 1319–1332.
[24] G. Longo, Space and time in the foundation of mathematics, or some challenges in the interaction with other sciences, Invited lecture at the First AMS/SMF meeting, Lyon, July 2001 (french version in Intellectica, 2003/1-2, n. 36–37).
[25] G. Longo, Some topologies for computations, Invited Lecture, in: Proceedings of Géométrie au XX siècle, 1930–2000, Hermann, Paris, 2004.
[26] G. Longo, The reasonable effectiveness of mathematics and its cognitive roots. In: L. Boi (ed.), Geometries of Nature, Living Systems and Human Cognition series in New Interactions of Mathematics with Natural Sciences and Humaties, World Scientific, 2005, pp. 351–382.
[27] G. Longo, Laplace, Turing and the imitation game impossible geometry: Randomness, determinism and programs in Turing's test, in: R. Epstein, G. Roberts, G. Beber (Eds.), Parsing the Turing Test, Springer, 2008.
[28] J. Myhill, J.C. Shepherdson, Effective operations on partial recursive functions, Zeitschrijt für Mathematische Logik und Grundlagen der Mathematik 1 (1955).
[29] Y. Moschovakis, Recursive metric spaces. Fundamenta. Mathematicae, LV, 1964.
[30] S.Yu. Pilyugin, Shadowing in Dynamical Systems, Springer, Berlin, 1999.
[31] M.B. Pour-El, J.I. Richards, Computability in analysis and physics, in: Perspectives in Mathematical Logic, Springer, Berlin, 1989.
[32] H. Rogers, Theory of Recursive Functions and Effective Computability, 1967.
[33] H. Rice, Classes of recursively enumerable sets and their decision problem, Transactions of the AMS 74 (1953).
[34] T. Sauer, Shadowing breakdown and large errors in dynamical simulations of physical systems, George Mason Univ., 2003, preprint.
[35] D.S. Scott, Outline of a mathematical theory of computation, in: 4th Ann. Princeton Conf. on Info. Sci. and Syst., 1970.
[36] D.S. Scott, Continuous lattices, in: Toposes, Algebraic Geometry and Logic, in: LNM, vol. 274, Springer-Verlag, Berlin, 1972.
[37] D.S. Scott, Data types as lattices, SIAM Journal of Computation 5 (1976) 522–587.
[38] J. Tappenden, Geometry and generality in Frege's philosophy of Arithmetic, Synthese 102 (3) (1995).
[39] A. Turing, On computable numbers with an application to the entscheidungsproblem, Proceedings of the London Mathematical Society 42 (1936).
[40] A. Turing, Computing machines and intelligence, Mind LIX (236) (1950) 433–460.
[41] A. Turing, The chemical basis of morphogenesis, Philolophical Transactions of the Royal Society B237 (1952) 37–72.
[42] K. Weihrauch, Computable Analysis, in: Texts in Theoretical Computer Science, Springer, Berlin, 2000.