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THE GEOMETRY OF MAXIMAL REPRESENTATIONS OF SURFACE GROUPS INTO $\mathrm{SO}_0(2,n)$

BRIAN COLLIER, NICOLAS THOLOZAN, AND JÉRÉMY TOULISSE

Abstract. In this paper, we study the geometric and dynamical properties of *maximal representations* of surface groups into Hermitian Lie groups of rank 2. Combining tools from Higgs bundle theory, the theory of Anosov representations, and pseudo-Riemannian geometry, we obtain various results of interest.

We prove that these representations are holonomies of certain geometric structures, recovering results of Guichard and Wienhard. We also prove that their length spectrum is uniformly bigger than that of a suitably chosen *Fuchsian* representation, extending a previous work of the second author. Finally, we show that these representations preserve a unique minimal surface in the symmetric space, extending a theorem of Labourie for Hitchin representations in rank 2.

Contents

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1. Introduction

In the past decades, two major theories have allowed many breakthroughs in the understanding of surface group representations.

On one side, non-abelian Hodge theory gives a bijective correspondence between conjugacy classes of representations of the fundamental group of a closed Riemann surface into a semi-simple Lie group and holomorphic objects on the Riemann surface called *Higgs bundles*. This theory, developed by Hitchin, Simpson, Corlette and many others, has proven very useful in describing the topology of character varieties of surface groups (see [Hit87], [Hit92] or [Got01]).

On the other side, Labourie showed that many surface group representations share a certain dynamical property called the *Anosov* property. This property has strong geometric and dynamical implications similar to the *quasi-Fuchsian* property for surface group representations in $\mathrm{PSL}(2,\mathbb{C})$.

A recent trend in the field is to try to link these two seemingly disparate theories (see for instance [AL15, Bar10, CL17]). Such links are far from being well-understood. For instance, there is no known Higgs bundle characterization of Anosov representations. The main obstacle is that finding the representation associated to a given Higgs bundle involves solving a highly transcendental system of PDEs called the *Higgs bundles equations*.

However, in some cases the Higgs bundle equations simplify, and one can hope to reach a reasonably good understanding of their solutions. These simplifications happen when the Higgs bundle is *cyclic*. Unfortunately, not every Higgs bundle is cyclic. Nevertheless, it turns out that restricting to cyclic Higgs bundles is enough to study representations into most Lie groups of real rank 2. This was used by Labourie [Lab17] to study Hitchin representations into $\mathrm{PSL}(3,\mathbb{R})$, $\mathrm{PSp}(4,\mathbb{R})$ and G_2, by the first author [Col16b] to study some maximal representations in $\mathrm{PSp}(4,\mathbb{R})$ and by the first author with Alessandrini [AC17] to study *all* maximal representations into $\mathrm{PSp}(4,\mathbb{R})$.

The goal of this paper is to derive from Higgs bundle theory several geometric properties of representations of surface groups into Hermitian Lie groups of rank 2. According to the work of Burger–Iozzi–Wienhard [BIW10], it is enough to restrict to representations into the Lie groups $\mathrm{SO}_0(2,n+1)$, $n \geq 1$ (see Remark 1.10).

Geometrization of maximal representations. Hitchin representations into split real Lie groups [Lab06] and maximal representations into Hermitian Lie groups [BILW05] are two important families of Anosov representations. One very nice feature of Anosov representations is that they are holonomies of certain geometric structures on closed manifolds. More precisely, for every Anosov representation ρ of a hyperbolic group Γ into a semi-simple Lie group G, Guichard and Wienhard

[GW12] construct a ρ-invariant open domain Ω in a certain *flag manifold* G/P on which $\rho(\Gamma)$ acts properly discontinuously and co-compactly.

In our setting, their result can be reformulated as follows. Let $\mathbb{R}^{2,n+1}$ denote the vector space $\mathbb{R}^{n+3}$ with the quadratic form

$$\mathbf{q}(\mathbf{x}) = x_1^2 + x_2^2 - x_3^2 - \ldots - x_{n+3}^2 \ .$$

We denote by $\mathbf{Ein}^{1,n}$ the space of isotropic lines in $\mathbb{R}^{2,n+1}$ and by $\mathbf{Pho}(\mathbb{R}^{2,n+1})$ the space of *photons* in $\mathbf{Ein}^{1,n}$ or, equivalently, of totally isotropic planes in $\mathbb{R}^{2,n+1}$. By Witt's theorem, $\mathrm{SO}_0(2,n+1)$ acts transitively on both $\mathbf{Ein}^{1,n}$ and $\mathbf{Pho}(\mathbb{R}^{2,n+1})$.

Theorem 1.1 (Guichard–Wienhard [GW12])**.** *Let Γ be the fundamental group of a closed oriented surface Σ of genus at least two. If $\rho : \Gamma \to \mathrm{SO}_0(2,n+1)$ is a maximal representation ($n \geq 2$), then there exists an open domain Ω_ρ in $\mathbf{Pho}(\mathbb{R}^{2,n+1})$ on which Γ acts properly discontinuously and co-compactly via ρ.*

In particular, the representation ρ is the holonomy of a *photon structure* on the closed manifold $\rho(\Gamma)\backslash\Omega_\rho$ (see Definition 4.10). One drawback of the construction of Guichard–Wienhard is that it a priori gives neither the topology of the domain Ω_ρ nor the topology of its quotient by $\rho(\Gamma)$. In forthcoming work [GW17a], a very clever – but very indirect – argument is used to describe this topology in the case of Hitchin representations in $\mathrm{SL}(2n,\mathbb{R})$. In an earlier paper, they focus on Hitchin representations into $\mathrm{SO}_0(2,3)$[1] and give a more explicit parametrization of (the two connected components of) Ω_ρ by triples of distinct points in $\mathbb{R}\mathbf{P}^1$, thus identifying $\rho(\Gamma)\backslash\Omega_\rho$ with the unit tangent bundle of Σ. In this parametrization, however, the circle bundle structure of the manifold is not apparent.

Here, we will construct photon structures on certain fiber bundles over Σ with holonomy any prescribed maximal representation in $\mathrm{SO}_0(2,n+1)$ in such a way that the fibers are "geometric". We will show that these photon structures coincide with the Guichard–Wienhard structures, and thus describe the topology of Guichard–Wienhard's manifolds in this setting.

Theorem 1.
Let Γ be the fundamental group of a closed oriented surface Σ of genus at least two. If $\rho : \Gamma \to \mathrm{SO}_0(2,n+1)$ is a maximal representation ($n \geq 2$), then there exists a fiber bundle $\pi : M \to \Sigma$ with fibers diffeomorphic to $\mathrm{O}(n)/\mathrm{O}(n-2)$, and a $\mathbf{Pho}(\mathbb{R}^{2,n+1})$-structure on M with holonomy $\rho \circ \pi_*$. Moreover, the developing map of this photon structure induces an isomorphism from each fiber of π to a copy of $\mathbf{Pho}(\mathbb{R}^{2,n}) \subset \mathbf{Pho}(\mathbb{R}^{2,n+1})$.

Conversely, if $\pi : M \to \Sigma$ is a fiber bundle with fibers diffeomorphic to $\mathrm{O}(n)/\mathrm{O}(n-2)$, then any photon structure on M whose developing map induces an isomorphism from each fiber of π to a copy of $\mathbf{Pho}(\mathbb{R}^{2,n}) \subset \mathbf{Pho}(\mathbb{R}^{2,n+1})$ has holonomy $\rho \circ \pi_*$, where $\rho : \Gamma \to \mathrm{SO}_0(2,n+1)$ is a maximal representation.

Corollary 2.
The manifold $\rho(\Gamma)\backslash\Omega_\rho$ in Guichard–Wienhard's Theorem 1.1 is diffeomorphic to a $\mathrm{O}(n)/\mathrm{O}(n-2)$-bundle over Σ.

[1] To be more accurate, Guichard and Wienhard study Hitchin representations into $\mathrm{PSL}(4,\mathbb{R})$ and in particular in $\mathrm{PSp}(4,\mathbb{R})$, and their action on the projective space $\mathbb{R}\mathbf{P}^3$. By a low dimension exceptional isomorphism, $\mathrm{PSp}(4,\mathbb{R})$ is isomorphic to $\mathrm{SO}_0(2,3)$ and $\mathbb{R}\mathbf{P}^3$ identifies (as a $\mathrm{PSp}(4,\mathbb{R})$-homogeneous space) with $\mathbf{Pho}(\mathbb{R}^{2,3})$.

Remark 1.2. The proof of Theorem 1 in Section 4 gives additional information on the topology of the fiber bundle M, which depends on certain topological invariants of the representation ρ.

Hitchin representations into $\mathrm{SO}_0(2,3)$ are the special class of maximal representations that also have a Guichard–Wienhard domain of discontinuity in $\mathbf{Ein}^{1,2}$. In a manner similar to [GW08], this domain can be parametrized by triples of distinct points in $\mathbb{R}\mathbf{P}^1$ so that its quotient by $\rho(\Gamma)$ is homeomorphic to the unit tangent bundle to Σ. Here, we recover this $\mathbf{Ein}^{1,2}$ structure (referred to as a conformally flat Lorentz structure) on the unit tangent bundle to Σ in such a way that the fibers are "geometric":

Theorem 3.
Let Γ be the fundamental group of a closed oriented surface Σ of genus at least two. Let $T^1\Sigma$ denote the unit tangent bundle to Σ and $\pi : T^1\Sigma \to \Sigma$ the bundle projection. If $\rho : \Gamma \to \mathrm{SO}_0(2,3)$ is a Hitchin representation, then there exists a $\mathbf{Ein}^{1,2}$-structure on $T^1\Sigma$ with holonomy $\rho \circ \pi_*$. Moreover, the developing map of this $\mathbf{Ein}^{1,2}$-structure induces an isomorphism from each fiber of π to a copy of $\mathbf{Ein}^{1,0} \subset \mathbf{Ein}^{1,2}$.

For the group $\mathrm{SO}_0(2,2)$, Alessandrini and Li [AL15] used Higgs bundle techniques to construct anti-de Sitter structures on circle bundles over Σ, recovering a result of Salein and Guéritaud-Kassel [Sal00, GK17].

Length spectrum of maximal representations in rank 2. Some Anosov representations of surface groups, such as Hitchin representations into real split Lie groups or maximal representations into Hermitian Lie groups, have the additional property of forming connected components of the whole space of representations. There have been several attempts to propose a unifying characterization of these representations (see [MZ16] and [GW17b]). Note that quasi-Fuchsian representations into $\mathrm{PSL}(2,\mathbb{C})$ do not form components; indeed, they can be continuously deformed into representations with non-discrete image.

The property of lying in a connected component consisting entirely of Anosov representations seems to be related to certain geometric controls of the representation "from below" such as an upper bound on the entropy or a *collar lemma.* To be more precise, let us introduce the *length spectrum* of a representation.

Definition 1.3. Let ρ be a representation of Γ into $\mathrm{SL}(n,\mathbb{R})$, $n \geq 2$. Let $[\Gamma]$ denote the set of conjugacy classes in Γ. The *length spectrum* of ρ is the function

$$\begin{array}{rccl} L_\rho : & [\Gamma] & \to & \mathbb{R}_+ \\ & \gamma & \mapsto & \frac{1}{2}\log\left|\frac{\lambda_1(\rho(\gamma))}{\lambda_n(\rho(\gamma))}\right| \end{array},$$

where $\lambda_1(A)$ and $\lambda_n(A)$ denote the complex eigenvalues of A with highest and lowest modulus respectively.

Remark 1.4. Since the eigenvalues of matrices in $\mathrm{SO}_0(2,n+1) \subset \mathrm{SL}(n+3,\mathbb{R})$ are preserved by the involution $A \mapsto A^{-1}$, the above definition simplifies to

$$L_\rho(\gamma) = \log|\lambda_1(\gamma)|$$

for representations into $\mathrm{SO}_0(2,n+1)$.

The length spectrum of a representation captures many of its algebraic, geometric and dynamical properties. Several results suggest that the length spectra of Hitchin and maximal representations are somehow always "bigger" than that of a Fuchsian representation. The first of these results deals with the "average behavior" of the length spectrum.

Definition 1.5. Let ρ be a representation of Γ into $\mathrm{SL}(n,\mathbb{R})$. The *entropy* of ρ is the number

$$h(\rho) = \limsup_{R\to+\infty} \frac{1}{R} \log \sharp\{\gamma \in [\Gamma] \mid L_\rho(\gamma) \leq R\} \ .$$

Theorem 1.6 (Potrie–Sambarino [PS14])**.** *If $\rho : \Gamma \to \mathrm{SL}(n,\mathbb{R})$ is a Hitchin representation, then*

$$h(\rho) \leq \frac{2}{n-1} \ ,$$

with equality if and only if ρ is conjugate to $m_{irr} \circ j$, where $j : \Gamma \to \mathrm{SL}(2,\mathbb{R})$ is a Fuchsian representation and $m_{irr} : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$ is the irreducible representation.

Another "geometric control" on Hitchin representations is a generalization of the classical *collar lemma* for Fuchsian representations. It roughly says that, if γ and η are two essentially intersecting curves on Σ, then $L_\rho(\gamma)$ and $L_\rho(\eta)$ cannot both be small. Such a collar lemma was obtained by Lee and Zhang for Hitchin representations into $\mathrm{SL}(n,\mathbb{R})$ [LZ14] and by Burger and Pozzetti [BP15] for maximal representations into $\mathrm{Sp}(2n,\mathbb{R})$. More precisely, they prove:

Theorem 1.7. *There exists a constant C such that, for any γ and η in $[\Gamma]$ represented by essentially intersecting curves on Σ and for any Hitchin (resp. maximal) representation ρ of Γ into $\mathrm{SL}(n,\mathbb{R})$ (resp. $\mathrm{Sp}(2n,\mathbb{R})$), one has*

$$\left(e^{L_\rho(\gamma)} - 1\right) \cdot \left(e^{L_\rho(\eta)} - 1\right) \geq C \ .$$

Motivated by a question of Zhang, the second author proved a stronger statement for Hitchin representations into $\mathrm{SL}(3,\mathbb{R})$ which implies both results above:

Theorem 1.8 (Tholozan, [Tho15])**.** *If $\rho : \Gamma \to \mathrm{SL}(3,\mathbb{R})$ is a Hitchin representation, then there exists a Fuchsian representation $j : \Gamma \to \mathrm{SL}(2,\mathbb{R})$ such that*

$$L_\rho \geq L_{m_{irr}\circ j} \ .$$

We will prove a similar statement for maximal representations into $\mathrm{SO}_0(2,n+1)$. A maximal representation $\rho : \Gamma \to \mathrm{SO}_0(2,n+1)$ is said to be *in the Fuchsian locus* if $\rho(\Gamma)$ preserves a copy of $\mathbb{R}^{2,1}$ in $\mathbb{R}^{2,n+1}$ (see Definition 2.7).

Theorem 4.
Let Γ be the fundamental group of a closed oriented surface Σ of genus at least two. If $\rho : \Gamma \to \mathrm{SO}_0(2,n+1)$ is a maximal representation ($n \geq 0$), then either ρ is in the Fuchsian locus, or there exists a Fuchsian representation $j : \Gamma \to \mathrm{SO}_0(2,1)$ and a $\lambda > 1$ such that

$$L_\rho \geq \lambda L_j \ .$$

As a direct consequence of the fact that Fuchsian representations into $\mathrm{SO}_0(2,1)$ have entropy 1, we obtain the following:

Corollary 5.
Let Γ be the fundamental group of a closed oriented surface Σ of genus at least two. If $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ is a maximal representation ($n \geq 0$), then the entropy $h(\rho)$ satisfies

$$h(\rho) \leq 1$$

with equality if and only if ρ is in the Fuchsian locus.

As a direct consequence of Theorem 4 and Keen's collar lemma [Kee74], we can also deduce a sharp collar lemma for maximal representations into $\mathrm{SO}_0(2, n+1)$:

Corollary 6.
Let Γ be the fundamental group of a closed oriented surface Σ of genus at least two and $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ be a maximal representation. If γ and η are two elements in $[\Gamma]$ represented by essentially intersecting curves on Σ, then

$$\sinh\left(\frac{L_\rho(\gamma)}{2}\right) \cdot \sinh\left(\frac{L_\rho(\eta)}{2}\right) > 1\ .$$

Labourie's conjecture for maximal representations in rank 2. A drawback of non-abelian Hodge theory is that it parameterizes representations of a surface group in a way that depends on the choice of a complex structure on the surface. In particular, such parameterizations do not have a natural action of the mapping class group of Σ. One would overcome this issue by finding a canonical way to associate to a given surface group representation a complex structure on the surface. To this intent, Labourie [Lab08] suggested the following approach.

Let $\mathcal{T}(\Sigma)$ denote the Teichmüller space of marked complex structures on Σ. For each reductive representation ρ of Γ into a semi-simple Lie group G, one can associate a functional on $\mathcal{T}(\Sigma)$ called the *energy functional*.

Definition 1.9. The *energy functional* $\mathbf{E}_\rho$ is the function that associates to a complex structure J on Σ the energy of the ρ-equivariant harmonic map from $(\widetilde{\Sigma}, J)$ to the Riemannian symmetric space G/K.

The existence of such an equivariant harmonic map was proven by Corlette [Cor88]. By a theorem of Sacks-Uhlenbeck and Schoen-Yau [SU77, SY79], J is a critical point of $\mathbf{E}_\rho$ if and only if the ρ-equivariant harmonic map from $(\widetilde{\Sigma}, J)$ to G/K is weakly conformal or, equivalently, if its image is a branched minimal surface in G/K. Labourie showed in [Lab08] that if the representation ρ is Anosov, then its energy functional is proper, and thus admits a critical point. He conjectured that, for Hitchin representations, this critical point is unique.

Conjecture (Labourie)**.** *Let Γ be the fundamental group of a closed oriented surface Σ of genus at least two. If ρ is a Hitchin representation of Γ into a real split Lie group G, then there is a unique complex structure $J \in \mathcal{T}(\Sigma)$ on Σ such that the ρ-equivariant harmonic map from $(\widetilde{\Sigma}, J)$ to G/K is weakly conformal.*

Labourie's conjecture was proven independently by Loftin [Lof01] and Labourie [Lab07] for $G = \mathrm{SL}(3, \mathbb{R})$, and then recently by Labourie [Lab17] for other split real Lie groups of rank 2 (namely, $\mathrm{PSp}(4, \mathbb{R})$ and G_2). Using the same strategy as Labourie, this was generalized by Alessandrini and the first author [Col16b, AC17] to all maximal representations into $\mathrm{PSp}(4, \mathbb{R})$. Here we give a new proof of their result and extend it to any Hermitian Lie group of rank 2.

THEOREM 7.
Let Γ be the fundamental group of a closed oriented surface Σ of genus at least two. If ρ is a maximal representation of Γ into a Hermitian Lie group G of rank 2, then there is a unique complex structure $J \in \mathcal{T}(\Sigma)$ such that the ρ-equivariant harmonic map from $(\widetilde{\Sigma}, J)$ to G/K is conformal. Moreover, this conformal harmonic map is an embedding.

Remark 1.10. Theorem 7 reduces to a theorem concerning maximal representations into $\mathrm{SO}_0(2,n)$. Indeed, the Hermitian Lie groups of rank 2 are (up to a cover): $\mathrm{PU}(1,n)\times\mathrm{PU}(1,n)$, $\mathrm{PSp}(4,\mathbb{R})$, $\mathrm{PU}(2,n)$ and $\mathrm{SO}_0(2,n)$ ($n \geq 5$). By [Tol89], maximal representations into $\mathrm{PU}(1,n)\times\mathrm{PU}(1,n)$ are conjugate to maximal representations into $\mathrm{P}(\mathrm{U}(1,1)\times\mathrm{U}(n-1))\times\mathrm{P}(\mathrm{U}(1,1)\times\mathrm{U}(n-1))$. By [BIW10], maximal representations into $\mathrm{PU}(2,n)$ are all conjugate to maximal representations into $\mathrm{P}(\mathrm{U}(2,2)\times U(n-2))$. Finally, $\mathrm{PU}(1,1)\times\mathrm{PU}(1,1)$ is isomorphic to $\mathrm{PSO}_0(2,2)$, $\mathrm{PSp}(4,\mathbb{R})$ is isomorphic to $\mathrm{SO}_0(2,3)$ and $\mathrm{PU}(2,2)$ is isomorphic to $\mathrm{PSO}_0(2,4)$.

Note that Labourie's conjecture does not hold for quasi-Fuchsian representations. Indeed, Huang and Wang [HW15] constructed quasi-Fuchsian manifolds containing arbitrarily many minimal surfaces. The conjecture seems to be related to the property of lying in a connected component of Anosov representations.

Maximal surfaces in $\mathbb{H}^{2,n}$ and strategy of the proof. Let $\mathbb{H}^{2,n}$ be the space of negative definite lines in $\mathbb{R}^{2,n+1}$. The space $\mathbb{H}^{2,n}$ is an open domain in $\mathbb{R}\mathbf{P}^{n+2}$ on which $\mathrm{SO}_0(2,n+1)$ acts transitively, preserving a pseudo-Riemannian metric of signature $(2,n)$ with constant sectional curvature -1. The boundary of $\mathbb{H}^{2,n}$ in $\mathbb{R}\mathbf{P}^{n+2}$ is the space $\mathbf{Ein}^{1,n}$. The cornerstone of all the above results will be the following theorem:

THEOREM 8.
Let Γ be the fundamental group of a closed oriented surface Σ of genus at least two. If $\rho : \Gamma \to \mathrm{SO}_0(2,n+1)$ is a maximal representation, then there exists a unique ρ-equivariant maximal space-like embedding of the universal cover of Σ into $\mathbb{H}^{2,n}$.

This theorem generalizes a well-known result of existence of maximal surfaces in some anti-de Sitter 3-manifolds. More precisely, for $n = 1$, maximal representations are exactly the holonomies of globally hyperbolic Cauchy-compact anti-de Sitter 3-manifolds (see [Mes07]). In this particular case, our theorem is due to Barbot, Béguin and Zeghib [BBZ03] (see also [Tou16] for the case with cone singularities).

The existence part of Theorem 8 will be proven in Section 3 using Higgs bundle theory. More precisely, we will see that, given a maximal representation $\rho : \Gamma \to \mathrm{SO}_0(2,n+1)$, any critical point of the energy functional $\mathbf{E}_\rho$ gives rise to a ρ-equivariant maximal space-like embedding of $\widetilde{\Sigma}$ with the same conformal structure. The uniqueness part of Theorem 8 will then directly imply Theorem 7. Our proof will use the pseudo-Riemannian geometry of $\mathbb{H}^{2,n}$ in a manner similar to [BS10]. Note that in a recent paper [DGK17], Danciger, Guéritaud and Kassel also use the geometry of the pseudo-hyperbolic space to understand special properties of Anosov representations.

We show in Subsection 3.4 that the ρ-equivariant minimal surface in the Riemannian symmetric space is the Gauss map of the maximal surface in $\mathbb{H}^{2,n}$. In the case $n = 1$, this interpretation recovers the equivalence between the existence of a unique maximal surface in globally hyperbolic anti-de Sitter 3-manifolds and the result of Schoen [Sch93] giving the existence of a unique minimal Lagrangian diffeomorphism isotopic to the identity between hyperbolic surfaces (the equivalence was proved in [KS07]).

Now, to each negative definite line $x \in \mathbb{H}^{2,n}$, one can associate a copy of $\mathbf{Pho}(\mathbb{R}^{2,n}) \subset \mathbf{Pho}(\mathbb{R}^{2,n+1})$ defined as the set of photons contained in $x^\perp$. Moreover, the copies of $\mathbf{Pho}(\mathbb{R}^{2,n})$ associated to such lines x and y are disjoint if and only if x and y are joined by a space-like geodesic. This remark allows us to construct a $\mathbf{Pho}(\mathbb{R}^{2,n+1})$ structure on a fiber bundle over Σ from the data of any ρ-equivariant space-like embedding of $\widetilde{\Sigma}$, and as a result, prove Theorem 1.

The $\mathbf{Ein}^{1,2}$-structures associated to Hitchin representations in $\mathrm{SO}_0(2,3)$ from Theorem 3 are constructed from the unique maximal space-like surface of Theorem 8 as follows. To each unit tangent vector v of the maximal space-like ρ-equivariant embedding of $\widetilde{\Sigma}$ in $\mathbb{H}^{2,2}$, one can associate a point in $\mathbf{Ein}^{1,2} = \partial_\infty \mathbb{H}^{2,2}$ by "following the geodesic determined by v to infinity". In this way, one obtains a ρ-equivariant map from $T^1\widetilde{\Sigma}$ to $\mathbf{Ein}^{1,2}$. Using a maximum principle involving the components of the solution to Higgs bundle equations, we will prove that this map is a local diffeomorphism. Note that this is specific to Hitchin representations and is not true for other maximal representations.

Finally, to prove Theorem 4, we introduce the length spectrum of the maximal ρ-equivariant embedding as an intermediate comparison. On the one hand, this length spectrum is larger than the length spectrum of the conformal metric of curvature -1 on the maximal surface, and on the other hand, it is less than the length spectrum of the representation ρ. This should be compared to [DT16] where Deroin and the second author prove that for any representation ρ into the isometry group of $\mathbb{H}^n$, there exists a Fuchsian representation j such that $L_j \geq L_\rho$. Here, both inequalities are reversed because of the pseudo-Riemannian geometry on $\mathbb{H}^{2,n}$.

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2. Maximal representations into $\mathrm{SO}_0(2, n+1)$

For the rest of the paper, Σ will be a closed surface of genus $g \geq 2$. We denote by Γ its fundamental group and by $\widetilde{\Sigma}$ its universal cover. Recall that the group Γ is *Gromov hyperbolic* and that its *boundary at infinity*, denoted by $\partial_\infty \Gamma$, is homeomorphic to a circle.

2.1. The Toledo invariant.

Let $\mathbb{R}^{2,n+1}$ denote the space $\mathbb{R}^{n+3}$ endowed with the quadratic form

$$\mathbf{q} : (x_1, \ldots, x_{n+3}) \mapsto x_1^2 + x_2^2 - x_3^2 - \ldots - x_{n+3}^2 \ .$$

The Lie group $\mathrm{SO}_0(2,n+1)$ is the identity component of the group of linear transformations of $\mathbb{R}^{2,n+1}$ preserving $\mathbf{q}$. Its subgroup $\mathrm{SO}(2)\times\mathrm{SO}(n+1)$ is a maximal compact subgroup.

To a representation $\rho:\Gamma\to\mathrm{SO}_0(2,n+1)$, one can associate a principal $\mathrm{SO}_0(2,n+1)$-bundle P_ρ whose total space is the quotient of $\widetilde{\Sigma}\times\mathrm{SO}_0(2,n+1)$ by the action of Γ by deck transformations:

$$\gamma\cdot(x,y)=(x\cdot\gamma^{-1},\rho(\gamma)y)\ .$$

Since the quotient of $\mathrm{SO}_0(2,n+1)$ by a maximal compact subgroup is contractible, this principal bundle admits a reduction of structure group to a principal $\mathrm{SO}(2)\times\mathrm{SO}(n+1)$-bundle B_ρ which is unique up to gauge equivalence. Finally, the quotient of B_ρ by the right action of $\mathrm{SO}(n+1)$ gives a principal $\mathrm{SO}(2)$-bundle M_ρ on Σ.

Definition 2.1. The *Toledo invariant* $\tau(\rho)$ of the representation ρ is the Euler class of the $\mathrm{SO}(2)$-bundle M_ρ.

The Toledo invariant is locally constant and invariant by conjugation. It thus defines a map

$$\tau:\ \mathrm{Rep}(\Gamma,\mathrm{SO}_0(2,n+1))\longrightarrow\mathbb{Z}\ ,$$

where $\mathrm{Rep}(\Gamma,\mathrm{SO}_0(2,n+1))$ denotes the set of conjugacy class of representations of Γ into $\mathrm{SO}_0(2,n+1)$). It is proven in [DT87] that the Toledo invariant satisfies the *Milnor–Wood inequality*:

Proposition 2.2. *For each representation $\rho:\Gamma\to\mathrm{SO}_0(2,n+1)$ the Toledo invariant satisfies*

$$|\tau(\rho)|\le 2g-2\ .$$

This leads to the following definition:

Definition 2.3. A representation $\rho:\Gamma\to\mathrm{SO}_0(2,n+1)$ is *maximal* if $|\tau(\rho)|=2g-2$.

2.2. **Maximal representations are Anosov.** The Toledo invariant and the notion of maximal representation can be defined more generally for representations of Γ into Hermitian Lie groups. In [BIW10], Burger, Iozzi and Wienhard study these representations. They prove in particular that for any Hermitian Lie group G *of tube type*, there exist maximal representations of Γ into G that have Zariski dense image. This applies in particular to maximal representations in $\mathrm{SO}_0(2,n+1)$.

In that same paper, they exhibit a very nice geometric property of maximal representations that was reinterpreted in [BILW05] as the *Anosov property* introduced independently by Labourie in [Lab06]. Here we describe one of the main consequences of their work in our setting.

Let $\mathbf{Ein}^{1,n}\subset\mathbb{R}\mathbf{P}^{n+2}$ denote the space of isotropic lines in $\mathbb{R}^{2,n+1}$. The group $\mathrm{SO}_0(2,n+1)$ acts transitively on $\mathbf{Ein}^{1,n}$ and preserves the conformal class of a pseudo-Riemannian metric of signature $(1,n)$. We will say that three isotropic lines $[e_1],[e_2]$ and $[e_3]$ in $\mathbf{Ein}^{1,n}$ are *in a space-like configuration* if the quadratic form $\mathbf{q}$ restricted to the vector space spanned by e_1, e_2 and e_3 has signature $(2,1)$.

Theorem 2.4 (Burger–Iozzi–Labourie–Wienhard, [BILW05])**.** *If $\rho:\Gamma\to\mathrm{SO}_0(2,n+1)$ is a maximal representation, then there is a unique ρ-equivariant continuous embedding*

$$\xi:\partial_\infty\Gamma\to\mathbf{Ein}^{1,n}\ .$$

Moreover, the image of ξ is a space-like curve*, meaning that the images of any three distinct points in $\partial_\infty\Gamma$ are in a space-like configuration.*

Note that the result of Burger, Iozzi, Labourie and Wienhard does not concern directly the case $G = \mathrm{SO}_0(2, n+1)$, but $G = \mathrm{SU}(p, q)$ and $G = \mathrm{Sp}(2n, \mathbb{R})$. However, as proved by Pozzetti and Hamlet [HP14], there exists a tight homomorphism $\iota : \mathrm{SO}_0(2, n+1) \to \mathrm{Sp}(2m, \mathbb{R})$ for some $m \in \mathbb{N}$. This property is sufficient to extend the result to the case of $\mathrm{SO}_0(2, n+1)$.

The Anosov property implies that maximal representations are *loxodromic*. In particular, the limit curve ξ can be reconstructed from the attracting and repelling eigenvectors of $\rho(\gamma)$ for $\gamma \in \Gamma$. More precisely, we have the following :

Corollary 2.5. *For every $\gamma \in \Gamma$, if γ_+ and γ_- denote the attracting and repelling fixed points of γ in $\partial_\infty\Gamma$, then, for $\lambda > 1$, $\xi(\gamma_+)$ and $\xi(\gamma_-)$ are the eigen-directions of $\rho(\gamma)$ for eigenvalues λ and λ^{-1} respectively. Moreover, the 2-plane spanned $\xi(\gamma_+)$ and $\xi(\gamma_-)$ is non-degenerate with respect to $\mathbf{q}$, and the restriction of $\rho(\gamma)$ to its orthogonal has spectral radius strictly less than λ.*

For $n = 0$, maximal representations into $\mathrm{SO}_0(2, 1)$ correspond to Fuchsian representations [Gol88b]. The isometric inclusion

$$\begin{array}{ccc} \mathbb{R}^{2,1} & \longrightarrow & \mathbb{R}^{2,n+1} \\ (x_1, x_2, x_3) & \longmapsto & (x_1, x_2, x_3, 0, \cdots, 0) \end{array}$$

defines an inclusion of $\mathrm{SO}_0(2, 1) \hookrightarrow \mathrm{SO}_0(2, n+1)$ which preserves the Toledo invariant. In particular, given a Fuchsian representation $\rho_{Fuch} : \Gamma \to \mathrm{SO}_0(2, 1)$, the $\mathrm{SO}_0(2, n+1)$-representation $j \circ \rho$ is maximal.

If $\alpha : \Gamma \to \mathrm{O}(n)$ is an orthogonal representation, let $\det(\alpha) : \Gamma \to \mathrm{O}(1)$ be the determinant representation. One can construct the representation

$$\rho_{Fuch} \otimes \det(\alpha) : \Gamma \to \mathrm{O}(2, 1),$$

obtained by twisting ρ_{Fuch} by $\det(\alpha)$. More precisely, $\rho_{Fuch} \otimes \det(\alpha)$ takes value in the index two subgroup of $\mathrm{O}(2, 1)$ preserving the orientation of space-like directions.

Proposition 2.6. *The maximal representation*

$$\rho = (\rho_{Fuch} \otimes \det(\alpha)) \oplus \alpha : \Gamma \to \mathrm{O}(2, n+1)$$

takes value in $\mathrm{SO}_0(2, n+1)$.

Proof. Because ρ_{Fuch} takes value in $\mathrm{SO}_0(2, 1)$, one can deform $\rho_{Fuch}(\gamma)$ to the identity in $\mathrm{SO}_0(2, 1)$ for any $\gamma \in \Gamma$. In particular, $\rho(\gamma)$ can be deformed to an element in $\mathrm{SO}(2) \times \mathrm{SO}(n+1) \subset \mathrm{SO}_0(2, n+1)$. □

Definition 2.7. A maximal representation $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ lies in the *Fuchsian locus* if it preserves a three dimensional linear subspace of $\mathbb{R}^{2,n+1}$ in restriction to which $\mathbf{q}$ has signature $(2, 1)$; equivalently,

$$\rho = \big(\rho_{Fuch} \otimes \det(\alpha)\big) \oplus \alpha$$

for $\rho_{Fuch} : \Gamma \to \mathrm{SO}_0(2, 1)$ a Fuchsian representation and $\alpha : \Gamma \to \mathrm{O}(n)$.

2.3. **Harmonic metrics and Higgs bundles.** We now recall the non-abelian Hodge correspondence between representations of Γ into $\mathrm{SO}_0(2,n+1)$ and $\mathrm{SO}_0(2,n+1)$-Higgs bundles. This correspondence holds for any real reductive Lie group G, but we will restrict the discussion to our group of interest.

When the surface Σ is endowed with a complex structure, we will denote the associate Riemann surface by X. The canonical bundle of X will be denoted by $\mathcal{K}$ and the trivial bundle will be denoted by $\mathcal{O}$. We also denote the Riemannian symmetric space of $\mathrm{SO}_0(2,n+1)$ by $\mathfrak{X}$, namely

$$\mathfrak{X} = \mathrm{SO}_0(2,n+1)/(\mathrm{SO}(2)\times\mathrm{SO}(n+1)).$$

We start by recalling the notion of a harmonic metric.

Definition 2.8. Let $\rho:\Gamma\to\mathrm{SO}_0(2,n+1)$ be a representation and let P_ρ be the associated flat $\mathrm{SO}_0(2,n+1)$-bundle. A *metric* on P_ρ is a reduction of structure group to $\mathrm{SO}(2)\times\mathrm{SO}(n+1)$. Equivalently, a metric is a ρ-equivariant map

$$\mathbf{h}_\rho : \ \widetilde{\Sigma} \longrightarrow \mathfrak{X}\ .$$

The differential $d\mathbf{h}_\rho$ of a metric $\mathbf{h}_\rho$ is a section of $T^*\widetilde{\Sigma}\otimes\mathbf{h}_\rho^*T\mathfrak{X}$. Given a metric g on Σ, one can define the norm $\|d\mathbf{h}_\rho\|$ of $d\mathbf{h}_\rho$ which, by equivariance of $\mathbf{h}_\rho$, is invariant under the action of Γ on $\widetilde{\Sigma}$ by deck transformations. In particular, $\|d\mathbf{h}_\rho\|$ descends to a function on Σ. The *energy* of $\mathbf{h}_\rho$ is the L^2-norm of $d\mathbf{h}_\rho$, namely:

$$\mathbf{E}(\mathbf{h}_\rho) = \int_\Sigma \|d\mathbf{h}_\rho\|^2 dv_g.$$

Note that the energy of $\mathbf{h}_\rho$ depends only on the conformal class of the metric g, and so, only on the Riemann surface structure X associated to g.

Definition 2.9. A metric $\mathbf{h}_\rho:\widetilde{X}\to\mathfrak{X}$ on P_ρ is *harmonic* if it is a critical point of the energy functional.

The complex structure on X and the Levi-Civita connection on $\mathfrak{X}$ induce a holomorphic structure $\nabla^{0,1}$ on the bundle $\big(T^*X\otimes\mathbf{h}_\rho^*T\mathfrak{X}\big)\otimes\mathbb{C}$. The following is classical (see [HW08, p. 425]):

Proposition 2.10. *A metric $\boldsymbol{h}_\rho:\widetilde{X}\to\mathfrak{X}$ is harmonic if and only if the $(1,0)$ part $\partial\boldsymbol{h}_\rho$ of $d\boldsymbol{h}_\rho$ is holomorphic, that is*

$$\nabla^{0,1}\partial\boldsymbol{h}_\rho = 0.$$

A representation $\rho:\Gamma\to\mathrm{SO}_0(2,n+1)$ is *completely reducible* if any $\rho(\Gamma)$-invariant subspace of $\mathbb{R}^{n+3}$ has a $\rho(\Gamma)$-invariant complement. For completely reducible representations, we have the following theorem.

Theorem 2.11 (Corlette [Cor88])**.** *A representation $\rho:\Gamma\to\mathrm{SO}_0(2,n+1)$ is completely reducible if and only if, for each Riemann surface structure X on $\mathbb{S}$, there exists a harmonic metric $\boldsymbol{h}_\rho:\widetilde{X}\to\mathfrak{X}$. Moreover, a harmonic metric is unique up to the action of the centralizer of ρ.*

Remark 2.12. In [BIW10], it is shown that all maximal representations are completely reducible and that the centralizer of a maximal representation in compact. Thus, for maximal representations there exists a unique harmonic metric.

For a completely reducible representation ρ, the energy of the harmonic metric $\mathbf{h}_\rho$ defines a function on the Teichmüller space $\mathcal{T}(\Sigma)$ of Σ

$$\begin{aligned} \mathbf{E}_\rho :\ \mathcal{T}(\Sigma) &\longrightarrow \mathbb{R}^{\geq 0} \\ X &\longmapsto \mathbf{E}(\mathbf{h}_\rho) \end{aligned} \quad . \tag{1}$$

The critical points of the energy are determined by the following.

Proposition 2.13 (Sacks-Uhlenbeck [SU77], Schoen-Yau [SY79])**.** *A harmonic metric $\boldsymbol{h}_\rho$ is a critical point of $\mathbf{E}_\rho$ if and only if it is weakly conformal, i.e. $tr(\partial \boldsymbol{h}_\rho \otimes \partial \boldsymbol{h}_\rho) = 0$. This is equivalent to $\boldsymbol{h}_\rho$ being a branched minimal immersion.*

For Anosov representations, Labourie has shown that the energy function (1) is smooth and proper, and so, has a critical point. As a corollary we have:

Proposition 2.14 (Labourie [Lab08])**.** *For each maximal representation there exists a Riemann surface structure on Σ for which the harmonic metric is weakly conformal.*

We now recall the notion of a Higgs bundle on a Riemann surface X.

Definition 2.15. An $\mathrm{SL}(n,\mathbb{C})$-Higgs bundle on X is a pair $(\mathcal{E},\Phi)$ where $\mathcal{E}$ is a rank n holomorphic vector bundle with $\Lambda^n\mathcal{E} = \mathcal{O}$ and $\Phi \in H^0(\mathrm{End}(\mathcal{E}) \otimes \mathcal{K})$ is a holomorphic endomorphism of $\mathcal{E}$ twisted by $\mathcal{K}$ with $\mathrm{tr}(\Phi) = 0$.

Higgs bundles were originally defined by Hitchin [Hit87] for the group $\mathrm{SL}(2,\mathbb{C})$ and generalized by Simpson [Sim88] for any complex semi-simple Lie group. More generally, Higgs bundles can be defined for real reductive Lie groups. For the group $\mathrm{SO}_0(2, n+1)$ the appropriate vector bundle definition is the following.

Definition 2.16. An $\mathrm{SO}_0(2, n+1)$-Higgs bundle over a Riemann surface X is a tuple $(\mathcal{U}, q_\mathcal{U}, \mathcal{V}, q_\mathcal{V}, \eta)$ where

- $\mathcal{U}$ and $\mathcal{V}$ are respectively rank 2 and rank (n+1) holomorphic vector bundles on X with trivial determinant and trivializations $\Lambda^2\mathcal{U} \cong \mathcal{O},\ \Lambda^{n+1}\mathcal{V} \cong \mathcal{O}$.
- $q_\mathcal{U}$ and $q_\mathcal{V}$ are non-degenerate holomorphic sections of $\mathrm{Sym}^2(\mathcal{U}^*)$ and $\mathrm{Sym}^2(\mathcal{V}^*)$,
- η is a holomorphic section of $\mathrm{Hom}(\mathcal{U},\mathcal{V}) \otimes \mathcal{K}$.

The non-degenerate sections $q_\mathcal{U}$ and $q_\mathcal{V}$ define holomorphic isomorphisms

$$q_\mathcal{U} : \mathcal{U} \to \mathcal{U}^* \qquad \text{and} \qquad q_\mathcal{V} : \mathcal{V} \to \mathcal{V}^* .$$

Given an $\mathrm{SO}_0(2, n+1)$-Higgs bundle $(\mathcal{U}, q_\mathcal{U}, \mathcal{V}, q_\mathcal{V}, \eta)$, we get an $\mathrm{SL}(n+3,\mathbb{C})$-Higgs bundle $(\mathcal{E},\Phi)$ by setting $\mathcal{E} = \mathcal{U} \oplus \mathcal{V}$ and

$$\Phi = \begin{pmatrix} 0 & \eta^\dagger \\ \eta & 0 \end{pmatrix} : \mathcal{U} \oplus \mathcal{V} \longrightarrow (\mathcal{U} \oplus \mathcal{V}) \otimes \mathcal{K}. \tag{2}$$

where $\eta^\dagger = q_\mathcal{U}^{-1} \circ \eta^T \circ q_\mathcal{V} \in H^0(\mathrm{Hom}(\mathcal{V},\mathcal{U}) \otimes \mathcal{K})$. Note that

$$\Phi^T \begin{pmatrix} q_\mathcal{U} & \\ & -q_\mathcal{V} \end{pmatrix} + \begin{pmatrix} q_\mathcal{U} & \\ & -q_\mathcal{V} \end{pmatrix} \Phi = 0.$$

Appropriate notions of poly-stability exist for G-Higgs bundles [GPGMiR09]. However, for our considerations, the following definition will suffice.

Definition 2.17. An $\mathrm{SL}(n,\mathbb{C})$-Higgs bundle $(\mathcal{E},\Phi)$ is stable if for all sub-bundles $\mathcal{F}\subset\mathcal{E}$ with $\Phi(\mathcal{F})\subset\mathcal{F}\otimes\mathcal{K}$ we have $\deg(\mathcal{F})<0$; $(\mathcal{E},\Phi)$ is called poly-stable if it is direct sum of stable $\mathrm{SL}(n_j,\mathbb{C})$-Higgs bundles. An $\mathrm{SO}_0(2,n+1)$-Higgs bundle is poly-stable if and only if the $\mathrm{SL}(n+3,\mathbb{C})$-Higgs bundle (2) is poly-stable.

From Higgs bundles to representations. Poly-stability is equivalent to existence of a Hermitian metric solving certain gauge theoretic equations which we refer to as the *Higgs bundle equations.* This was proven by Hitchin [Hit87] for $\mathrm{SL}(2,\mathbb{C})$ and Simpson [Sim88] for semi-simple complex Lie groups, see [GPGMiR09] for the statement for real reductive groups.

We say that a Hermitian metric h on $\mathcal{E}$ is *adapted* to the $\mathbb{C}$-bilinear symmetric form $q=q_{\mathcal{U}}\oplus -q_{\mathcal{V}}$ if $h(u,v)=q(u,\lambda(v))$ where $\lambda:\mathcal{E}\to\mathcal{E}$ is an anti-linear involution. In such a case, we say that λ is the involution associated to the metric h.

Theorem 2.18. *An $\mathrm{SO}_0(2,n+1)$-Higgs bundle $(\mathcal{U},q_{\mathcal{U}},\mathcal{V},q_{\mathcal{V}},\eta)$ is poly-stable if and only if there exist adapted Hermitian metrics $h_{\mathcal{U}}$ and $h_{\mathcal{V}}$ on $\mathcal{U}$ and $\mathcal{V}$ such that*

$$\begin{cases} F_{h_{\mathcal{U}}}+\eta^\dagger\wedge(\eta^\dagger)^{*_h}+\eta^{*_h}\wedge\eta=0\\ F_{h_{\mathcal{V}}}+\eta\wedge\eta^{*_h}+(\eta^\dagger)^{*_h}\wedge\eta^\dagger=0\end{cases}\tag{3}$$

Here $F_{h_{\mathcal{U}}}$ and $F_{h_{\mathcal{V}}}$ denote the curvature of the Chern connections of $h_{\mathcal{U}}$ and $h_{\mathcal{V}}$ and η^{*_h} denotes the Hermitian adjoint of η, i.e. $h_{\mathcal{V}}(u,\eta(v))=h_{\mathcal{U}}(\eta^{*_h}(u),v)$.

If $(h_{\mathcal{U}},h_{\mathcal{V}})$ solves the Higgs bundle equations (3), then the metric $h=h_{\mathcal{U}}\oplus h_{\mathcal{V}}$ on $\mathcal{E}=\mathcal{U}\oplus\mathcal{V}$ solves the $\mathrm{SL}(n+3,\mathbb{C})$-Higgs bundle equations

$$F_h+[\Phi,\Phi^{*h}]=0.$$

Given a solution $(h_{\mathcal{U}},h_{\mathcal{V}})$ to the Higgs bundle equations, the connection

$$\nabla=\begin{pmatrix}\nabla_{h_{\mathcal{U}}} & \\ & \nabla_{h_{\mathcal{V}}}\end{pmatrix}+\begin{pmatrix}0 & \eta^\dagger\\ \eta & 0\end{pmatrix}+\begin{pmatrix}0 & \eta^{*_h}\\ (\eta^\dagger)^{*_h} & 0\end{pmatrix}\tag{4}$$

is a *flat* connection on $\mathcal{E}=\mathcal{U}\oplus\mathcal{V}$. Moreover, if $\lambda_{\mathcal{U}}$ and $\lambda_{\mathcal{V}}$ are the associated involutions, $\lambda_{\mathcal{U}}\oplus\lambda_{\mathcal{V}}$ is preserved by ∇.

Denote the associated real bundle by E_∇. The orthogonal structure $q_{\mathcal{U}}\oplus -q_{\mathcal{V}}$ restricts to a ∇-parallel signature $(2,n+1)$ metric $g_{\mathcal{U}}\oplus g_{\mathcal{V}}$ on E_∇. The holonomy of ∇ gives a representation $\rho:\Gamma\to\mathrm{SO}_0(2,n+1)$ which is completely reducible.

From representations to Higgs bundles. Let (E_ρ,∇,g) be the flat rank $(n+3)$ vector bundle with signature $(2,n+1)$ metric g and flat connection ∇ associated to a representation $\rho:\Gamma\to\mathrm{SO}_0(2,n+1)$. A metric on E_ρ,

$$\mathbf{h}_\rho:\ \widetilde{\Sigma}\longrightarrow\mathfrak{X}$$

is equivalent to a splitting $E_\rho=U\oplus V$ where U is a rank 2 orthogonal bundle with $g_U=g|_U$ positive definite and V is a rank $(n+1)$-bundle with $-g_V=-g|_V$ positive definite. Moreover, the flat connection ∇ decomposes as

$$\nabla=\begin{pmatrix}\nabla_U & \\ & \nabla_V\end{pmatrix}+\begin{pmatrix} & \Psi^\dagger\\ \Psi & \end{pmatrix}\tag{5}$$

where ∇_U and ∇_V are connections on U and V such that g_U and g_V are covariantly constant, Ψ is a one form valued in the bundle $\mathrm{Hom}(U,V)$ and $\Psi^\dagger=g_U^{-1}\Psi^Tg_V$.

The one form $(\Psi+\Psi^\dagger)\in\Omega^1(\Sigma,\mathrm{Hom}(U,V)\oplus\mathrm{Hom}(V,U))$ is identified with the differential of the metric $\mathbf{h}_\rho$. If X is a Riemann surface structure on Σ, then the

Hermitian extension $h_U \oplus h_V$ of $g_U \oplus -g_V$ to the complexification of E_ρ defines a Hermitian metric. The complex linear extensions of ∇_U, ∇_V, Ψ and $\Psi^\dagger$ all decompose into $(1,0)$ and $(0,1)$ parts, and $\nabla_U^{0,1}$ and $\nabla_V^{0,1}$ define holomorphic structures. Writing $\nabla_{U,V}^{0,1}$ for the $(0,1)$-part of the connection on $\mathrm{Hom}(U,V)$ induced by the connections ∇_U and ∇_V, Proposition 2.10 reads:

Proposition 2.19. *A metric $\boldsymbol{h}_\rho : \widetilde{X} \to \mathfrak{X}$ is harmonic if and only if $\nabla_{U,V}^{0,1}\Psi^{1,0} = 0$, (or equivalently $\nabla_{V,U}^{0,1}(\Psi^\dagger)^{1,0}$).*

Given a harmonic metric $\mathbf{h}_\rho$, the Hermitian adjoints of $\Psi^{1,0}$ and $(\Psi^\dagger)^{0,1}$ are given by $(\Psi^{1,0})^* = (\Psi^\dagger)^{0,1}$ and $(\Psi^\dagger)^{1,0} = (\Psi^{0,1})^*$. With respect to a harmonic metric, the flatness equations $F_\nabla = 0$ decompose as

$$\left\{ \begin{array}{l} F_{\nabla_U} + \Psi^{1,0} \wedge (\Psi^{1,0})^* + ((\Psi^\dagger)^{1,0})^* \wedge (\Psi^\dagger)^{1,0} = 0 \\ F_{\nabla_V} + (\Psi^\dagger)^{1,0} \wedge ((\Psi^\dagger)^{1,0})^* + (\Psi^{1,0})^* \wedge \Psi^{1,0} = 0 \\ \nabla_{U,V}^{0,1}\Psi^{1,0} = 0 \end{array} \right. . \tag{6}$$

Note that setting $\Psi^{1,0} = \eta$, the Higgs bundle equations (3) are the same as the decomposition of the flatness equations (6) with respect to a harmonic metric. Thus, if $\mathcal{U}$ and $\mathcal{V}$ are the holomorphic bundles $(U \otimes \mathbb{C}, \nabla_U^{0,1})$ and $(V \otimes \mathbb{C}, \nabla_V^{0,1})$, then $(\mathcal{U}, q_\mathcal{U}, \mathcal{V}, q_\mathcal{V}, \Psi^{1,0})$ is a poly-stable $\mathrm{SO}_0(2, n+1)$-Higgs bundle, where $q_\mathcal{U}$ is the $\mathbb{C}$-linear extension of g_U to $U \otimes \mathbb{C}$ (similarly for $q_\mathcal{V}$).

Proposition 2.20. *Let $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ be a completely reducible representation and X be a Riemann surface structure on Σ. If $(\mathcal{U}, q_\mathcal{U}, \mathcal{V}, q_\mathcal{V}, \eta)$ is the Higgs bundle associated to ρ, then the harmonic metric $\boldsymbol{h}_\rho$ is a branched minimal immersion if and only if $tr(\eta \otimes \eta^\dagger) = 0$.*

Proof. The derivative of the harmonic metric is identified with the 1-form $\Psi + \Psi^\dagger$ from (5). By Proposition 2.13, $\mathbf{h}_\rho$ is a branched minimal immersion if and only if

$$\mathrm{tr}\left(\begin{pmatrix} 0 & (\Psi^\dagger)^{0,1} \\ \Psi^{0,1} & 0 \end{pmatrix}^2 \right) = 0.$$

This is equivalent to $\mathrm{tr}(\eta \otimes \eta^\dagger) = 0$. □

Definition 2.21. An $\mathrm{SO}_0(2, n+1)$-Higgs bundle $(\mathcal{U}, q_\mathcal{U}, \mathcal{V}, q_\mathcal{V}, \eta)$ will be called *conformal* if $\mathrm{tr}(\eta \otimes \eta^\dagger) = 0$.

2.4. **Maximal Higgs bundle parameterizations.** We now describe the Higgs bundles which give rise to maximal $\mathrm{SO}_0(2, n+1)$-representations.

Proposition 2.22. *The isomorphism class of a $\mathrm{SO}_0(2, n+1)$-Higgs bundle $(\mathcal{U}, q_\mathcal{U}, \mathcal{V}, q_\mathcal{V}, \eta)$ is determined by the data $(\mathcal{L}, \mathcal{V}, q_\mathcal{V}, \beta, \gamma)$ where $\mathcal{L}$ is a holomorphic line bundle on X, $\beta \in H^0(\mathcal{L} \otimes \mathcal{V} \otimes \mathcal{K})$ and $\gamma \in H^0(\mathcal{L}^{-1} \otimes \mathcal{V} \otimes \mathcal{K})$. Here $\mathcal{U} = \mathcal{L} \oplus \mathcal{L}^{-1}$, $q_\mathcal{U} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\eta = (\gamma, \beta) : \mathcal{L} \oplus \mathcal{L}^{-1} \to \mathcal{V} \otimes \mathcal{K}$. Moreover, if $(\mathcal{U}, q_\mathcal{U}, \mathcal{V}, q_\mathcal{V}, \eta)$ is poly-stable, then the Toledo invariant of the corresponding representation is the degree of $\mathcal{L}$.*

Proof. The group $\mathrm{SO}(2, \mathbb{C})$ is isomorphic to the set of 2×2 matrices A such that $A^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since the bundle $(\mathcal{U}, q_\mathcal{U})$ is the associated bundle of a holomorphic principal $\mathrm{SO}(2,\mathbb{C})$-bundle, up to isomorphism we have

$$(\mathcal{U},q_{\mathcal{U}})=\left(\mathcal{L}\oplus\mathcal{L}^{-1},\begin{pmatrix}0&1\\1&0\end{pmatrix}:\mathcal{L}\oplus\mathcal{L}^{-1}\to(\mathcal{L}\oplus\mathcal{L}^{-1})^*\right).$$

With respect to the splitting $\mathcal{U}=\mathcal{L}\oplus\mathcal{L}^{-1}$, the holomorphic section $\eta\in\mathrm{Hom}(\mathcal{U},\mathcal{V})\otimes\mathcal{K}$ decomposes as $\beta\oplus\gamma$ where $\beta\in\mathrm{Hom}(\mathcal{L}\otimes\mathcal{V}\otimes\mathcal{K})$ and $\gamma\in\mathrm{Hom}(\mathcal{L}^{-1}\otimes\mathcal{V}\otimes\mathcal{K})$. Since, the degree of $\mathcal{L}$ is the degree of the $\mathrm{SO}(2)$-bundle whose complexification is $\mathcal{U}$, the Toledo invariant of the associated representation is the degree of $\mathcal{L}$. □

Remark 2.23. The $\mathrm{SL}(n+3,\mathbb{C})$-Higgs bundle $(\mathcal{E},\Phi)$ associated to $(\mathcal{L},\mathcal{V},q_{\mathcal{V}},\beta,\gamma)$ is given by $\mathcal{E}=\mathcal{L}\oplus\mathcal{L}^{-1}\oplus\mathcal{V}$ and

$$\Phi=\begin{pmatrix}0&0&\beta^\dagger\\0&0&\gamma\dagger\\\gamma&\beta&0\end{pmatrix}:\mathcal{E}\to\mathcal{E}\otimes\mathcal{K}. \tag{7}$$

The Milnor-Wood inequality can be seen directly for poly-stable Higgs bundles.

Proposition 2.24. *If $(\mathcal{L},\mathcal{V},q_{\mathcal{V}},\beta,\gamma)$ is a poly-stable $\mathrm{SO}_0(2,n+1)$-Higgs bundle, then $\deg(\mathcal{L})\leq 2g-2$. Furthermore, if $\deg(\mathcal{L})=2g-2$, then*

- *$\mathcal{V}$ admits a $q_{\mathcal{V}}$-orthogonal decomposition $\mathcal{V}=\mathcal{I}\oplus\mathcal{V}_0$ where $\mathcal{V}_0$ is a holomorphic rank n bundle and $\mathcal{I}=\Lambda^n\mathcal{V}_0$ satisfies $\mathcal{I}^2=\mathcal{O}$.*
- *$\mathcal{L}\cong\mathcal{I}\mathcal{K}$*
- *$\gamma\cong\begin{pmatrix}1\\0\end{pmatrix}:\mathcal{I}\mathcal{K}\to\mathcal{I}\mathcal{K}\oplus\mathcal{V}_0\otimes\mathcal{K}$ and $\beta=\begin{pmatrix}q_2\\\beta_0\end{pmatrix}:\mathcal{K}^{-1}\mathcal{I}\to\mathcal{I}\mathcal{K}\oplus\mathcal{V}_0\otimes\mathcal{K}$ where $q_2\in H^0(\mathcal{K}^2)$ and $\beta_0\in H^0(\mathcal{K}\otimes\mathcal{I}\otimes\mathcal{V}_0)$.*

Proof. The poly-stable $\mathrm{SL}(n+3,\mathbb{C})$ Higgs bundle $(\mathcal{E},\Phi)$ associated to $(\mathcal{L},\mathcal{V},q_{\mathcal{V}},\beta,\gamma)$ has $\mathcal{E}=\mathcal{L}\oplus\mathcal{L}^{-1}\oplus\mathcal{V}$ and Φ is given by (7). If $\deg(\mathcal{L})>0$, then by poly-stability $\gamma\neq 0$. If the image of γ is isotropic, then we have a sequence

$$0\longrightarrow\mathcal{L}\mathcal{K}^{-1}\xrightarrow{\gamma}\ker(\gamma^\dagger)\longrightarrow\mathcal{V}\xrightarrow{\gamma^\dagger}\mathcal{L}^{-1}\mathcal{K}\longrightarrow 0\ .$$

Since $\deg(\ker(\gamma^\dagger))=\deg(\mathcal{L})-(2g-2)$ and $\mathcal{L}\oplus\ker(\gamma^\dagger)$ is an invariant sub-bundle, we have $\deg(\mathcal{L})\leq g-1$. Thus, for $\deg(\mathcal{L})>g-1$ the composition $\gamma\circ\gamma^\dagger$ is a non-zero element of $H^0((\mathcal{L}^{-1}\mathcal{K})^2)$, and we conclude $\deg(\mathcal{L})\leq 2g-2$.

If $\deg(\mathcal{L})=2g-2$, then $(\mathcal{L}^{-1}\mathcal{K})^2=\mathcal{O}$ and γ is nowhere vanishing. Set $\mathcal{I}=\mathcal{L}\mathcal{K}^{-1}$, then $\mathcal{L}=\mathcal{I}\mathcal{K}$ and $\mathcal{I}$ defines an orthogonal line sub-bundle of $\mathcal{V}$. Taking the $q_{\mathcal{V}}$-orthogonal complement of $\mathcal{I}$ gives a holomorphic decomposition $\mathcal{V}=\mathcal{I}\oplus(\mathcal{I})^\perp$. Since $\Lambda^{n+1}\mathcal{V}=\mathcal{O}$, we conclude $\mathcal{V}=\mathcal{I}\oplus\mathcal{V}_0$ where $\mathcal{I}=\Lambda^n\mathcal{V}_0$. Since the image of γ is identified with $\mathcal{I}$, we can take $\gamma\cong\begin{pmatrix}1\\0\end{pmatrix}:\mathcal{I}\mathcal{K}\to\mathcal{I}\mathcal{K}\oplus\mathcal{V}_0\otimes\mathcal{K}$. Finally, the holomorphic section β of $\mathrm{Hom}(\mathcal{I}\mathcal{K}^{-1},\mathcal{I}\oplus\mathcal{V}_0)\otimes\mathcal{K}$ decomposes as

$$\beta=q_2\oplus\beta_0$$

where q_2 is a holomorphic quadratic differential and $\beta_0\in H^0(\mathcal{V}_0\otimes\mathcal{I}\mathcal{K})$ □

Remark 2.25. Higgs bundles with $\deg(\mathcal{L})=2g-2$ will be called maximal Higgs bundles. They are determined by tuples $(\mathcal{V}_0,q_{\mathcal{V}_0},\beta_0,q_2)$ from Proposition 2.24.

Proposition 2.26. *If $\rho:\mathrm{Rep}^{max}(\Gamma,\mathrm{SO}_0(2,n+1))$ is a maximal representation, X is a Riemann surface structure on Σ and the Higgs bundle corresponding to ρ is defined by the data $(\mathcal{V}_0,q_{\mathcal{V}_0},\beta_0,q_2)$, then the harmonic metric is a minimal immersion if and only if the holomorphic quadratic differential q_2 vanishes.*

Proof. By Proposition 2.20, the harmonic metric associated to a poly-stable Higgs bundle $(\mathcal{U}, q_{\mathcal{U}}, \mathcal{V}, q_{\mathcal{V}}, \eta)$ is a branched minimal immersion if and only if $\operatorname{tr}(\eta \otimes \eta^\dagger) = 0$. For a maximal Higgs bundle determined by $(\mathcal{V}_0, q_{\mathcal{V}_0}, \beta_0, q_2)$

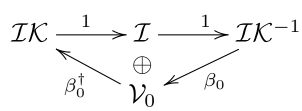
$$\eta = \begin{pmatrix} 1 & 0 \\ q_2 & \beta_0 \end{pmatrix} : \mathcal{IK} \oplus \mathcal{IK}^{-1} \to \mathcal{IK} \oplus \mathcal{V}_0 \mathcal{K} \quad \text{and} \quad \eta^\dagger = \begin{pmatrix} q_2 & \beta_0^\dagger \\ 1 & 0 \end{pmatrix} : \mathcal{I} \oplus \mathcal{V}_0 \to \mathcal{IK}^2 \oplus \mathcal{I}.$$

A computation shows $\operatorname{tr}(\eta \otimes \eta^\dagger) = 2q_2$, thus, by Proposition 2.20, the harmonic map is a branched minimal immersion if and only if $q_2 = 0$. Finally, $\eta + \eta^\dagger$ is nowhere vanishing, hence the branched minimal immersion is branch point free. □

Given a maximal representation $\rho \in \operatorname{Rep}(\Gamma, \mathrm{SO}_0(2, n+1))$, by Proposition 2.14, we can always find a Riemann surface structure in which the corresponding Higgs bundle is a maximal conformal Higgs bundle. A maximal conformal Higgs bundle is determined by $(\mathcal{V}_0, q_{\mathcal{V}_0}, \beta_0)$:

$$(\mathcal{U}, q_{\mathcal{U}}, \mathcal{V}, q_{\mathcal{V}}, \eta) = \left(\mathcal{IK} \oplus \mathcal{IK}^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathcal{I} \oplus \mathcal{V}_0, \begin{pmatrix} 1 & 0 \\ 0 & q_{\mathcal{V}_0} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \beta_0 \end{pmatrix} \right).$$

The associated $\mathrm{SL}(n+3, \mathbb{C})$-Higgs bundle will be represented schematically by



where the arrows represent the Higgs field and we omit the tensor product by $\mathcal{K}$. Such a Higgs bundle is an example of a *cyclic* Higgs bundle.

Definition 2.27. An $\mathrm{SL}(n, \mathbb{C})$-Higgs bundle $(\mathcal{E}, \Phi)$ is called *cyclic of order* k if there is a holomorphic splitting $\mathcal{E} = \mathcal{E}_1 \oplus \ldots \oplus \mathcal{E}_k$ such that Φ maps $\mathcal{E}_i$ into $\mathcal{E}_{i+1} \otimes \mathcal{K}$ (for $i < k$) and $\mathcal{E}_k$ to $\mathcal{E}_1 \otimes \mathcal{K}$.

Proposition 2.28 (Simpson, [Sim09])**.** *If the Higgs bundle (E, Φ) is cyclic of order k, then the cyclic splitting of Φ is orthogonal with respect to the Hermitian metric which solves the Higgs equations $F_H + [\Phi, \Phi^*] = 0$.*

The symmetries of the solution metrics (3) and Proposition 2.28 give a further simplification of the Higgs bundle equations for maximal conformal $\mathrm{SO}_0(2, n+1)$-Higgs bundles.

Proposition 2.29. *For a poly-stable maximal conformal $\mathrm{SO}_0(2, n+1)$-Higgs bundle determined by $(\mathcal{V}_0, q_{\mathcal{V}_0}, \beta_0)$, if $(h_{\mathcal{U}}, h_{\mathcal{V}})$ solves the Higgs bundle equations (3), then*

- $h_{\mathcal{U}} = \begin{pmatrix} h_{\mathcal{IK}} & \\ & h_{\mathcal{IK}}^{-1} \end{pmatrix}$ *where $h_{\mathcal{IK}}$ is a metric on $\mathcal{IK}$ and $h_{\mathcal{IK}}^{-1}$ is the induced metric on $\mathcal{IK}^{-1}$*
- $h_{\mathcal{V}} = \begin{pmatrix} h_{\mathcal{I}} & \\ & h_{\mathcal{V}_0} \end{pmatrix}$ *where $h_{\mathcal{I}}$ is a flat metric on $\mathcal{I}$ and $h_{\mathcal{V}_0}$ is a metric on $\mathcal{V}_0$ adapted to $q_{\mathcal{V}_0}$.*

Furthermore, the Higgs bundle equations (3) simplify as

$$\left\{ \begin{array}{l} F_{h_{\mathcal{IK}}} + \beta_0^\dagger \wedge (\beta_0^\dagger)^{*_h} + 1^{*_h} \wedge 1 = 0 \\ F_{\mathcal{V}_0} + \beta_0 \wedge \beta_0^{*_h} + (\beta_0^\dagger)^{*_h} \wedge \beta_0^\dagger = 0 \end{array} \right. \tag{8}$$

Proof. Because $h_{\mathcal{U}}$ is adapted to $q_{\mathcal{U}}$, $h_{\mathcal{U}} = \begin{pmatrix} h_{\mathcal{IK}} & \\ & h_{\mathcal{IK}}^{-1} \end{pmatrix}$ where $h_{\mathcal{IK}}$ is a metric on $\mathcal{IK}$ and $h_{\mathcal{IK}}^{-1}$ is the induced metric on $\mathcal{IK}^{-1}$. The splitting of $h_{\mathcal{V}}$ follows from Proposition 2.28. The Higgs bundle equations (3) with $\eta = \begin{pmatrix} 1 & 0 \\ 0 & \beta_0 \end{pmatrix}$ and $h_{\mathcal{U}} = h_{\mathcal{IK}} \oplus h_{\mathcal{IK}}^{-1}$ and $h_{\mathcal{V}} = h_{\mathcal{I}} \oplus h_{\mathcal{V}_0}$ simplify to

$$\begin{cases} F_{h_{\mathcal{IK}}} + \beta_0^\dagger \wedge (\beta_0^\dagger)^{*_h} + 1^{*_h} \wedge 1 = 0 \\ F_{h_{\mathcal{IK}}^{-1}} + 1 \wedge 1^{*_h} + \beta_0^{*_h} \wedge \beta_0 = 0 \\ F_{h_{\mathcal{I}}} + 1 \wedge 1^{*_h} + 1^{*_h} \wedge 1 = 0 \\ F_{\mathcal{V}_0} + \beta_0 \wedge \beta_0^{*_h} + (\beta_0^\dagger)^{*_h} \wedge \beta_0^\dagger = 0 \end{cases}$$

Note that the first two equations are the same and the third equation implies the metric $h_{\mathcal{I}}$ is flat. □

For a maximal poly-stable conformal $\mathrm{SO}_0(2,n+1)$-Higgs bundle determined by $(\mathcal{V}_0, q_{\mathcal{V}_0}, \beta_0)$, the associated flat bundle $E_\rho \subset (\mathcal{IK} \oplus \mathcal{IK}^{-1} \oplus \mathcal{I} \oplus \mathcal{V}_0)$ is the fixed point locus of the associated anti-linear involution $\lambda : \mathcal{E} \to \mathcal{E}$, that is the involution defined by the equation $h(u,v) = q(u, \lambda(v))$.

In the splitting $\mathcal{E} = \mathcal{IK} \oplus \mathcal{IK}^{-1} \oplus \mathcal{I} \oplus \mathcal{V}_0$, the $\mathbb{C}$-bilinear form q is given by

$$q = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & -1 & \\ & & & -q_{\mathcal{V}_0} \end{pmatrix}.$$

By Proposition 2.29, the previous splitting is orthogonal with respect to the Hermitian metric solving the Higgs bundle equations. In particular, one easily checks that the associated involution $\lambda : \mathcal{IK} \oplus \mathcal{IK}^{-1} \oplus \mathcal{I} \oplus \mathcal{V}_0 \to \mathcal{IK} \oplus \mathcal{IK}^{-1} \oplus \mathcal{I} \oplus \mathcal{V}_0$ is written

$$\lambda = \begin{pmatrix} & h_{\mathcal{IK}}^{-1} & & \\ h_{\mathcal{IK}} & & & \\ & & -h_{\mathcal{I}} & \\ & & & -\lambda_{\mathcal{V}_0} \end{pmatrix},$$

where $h_{\mathcal{IK}}(u)$ is the anti-linear map defined by $h_{\mathcal{IK}}(u).v = h_{\mathcal{IK}}(u,v)$.

Thus the flat bundle $E_\rho = U \oplus V$ of a maximal representation decomposes further. This decomposition will play an essential role in the rest of the paper.

Theorem 2.30. *The flat bundle associated to a poly-stable maximal conformal* $\mathrm{SO}_0(2,n+1)$*-Higgs bundle determined by* $(\mathcal{V}_0, q_{\mathcal{V}_0}, \beta_0)$ *decomposes as*

$$E_\rho = U \oplus \ell \oplus V_0$$

where $U \subset \mathcal{U}$ *is a positive definite rank two sub-bundle,* $\ell \subset \mathcal{I}$ *is a negative definite line sub-bundle and* $V_0 \subset \mathcal{V}_0$ *is a negative definite rank* n *bundle. In this splitting the flat connection is given by*

$$\nabla = \begin{pmatrix} \nabla_{h_{\mathcal{U}}} & 1 + 1^{*_h} & \beta_0^\dagger + \beta_0^{*_h} \\ 1 + 1^{*_h} & \nabla_{h_{\mathcal{I}}} & 0 \\ \beta_0 + (\beta^\dagger)^{*_h} & 0 & \nabla_{h_{\mathcal{V}_0}} \end{pmatrix}$$

From now on we will only consider poly-stable maximal $\mathrm{SO}_0(2,n+1)$-Higgs bundles. For notational convenience, we will drop the subscript 0 and write the decomposition of the flat bundle E_ρ as $E_\rho = U \oplus \ell \oplus V$.

2.5. **Connected components of maximal representations.** Given a maximal $\mathrm{SO}_0(2,n+1)$-Higgs bundle

$$\begin{array}{ccc} \mathcal{IK} \xrightarrow{1} & \mathcal{I} & \xrightarrow{1} \mathcal{IK}^{-1} \\ \nwarrow_{\beta^\dagger} & \oplus & \swarrow_{\beta} \\ & \mathcal{V} & \end{array}, \tag{9}$$

the Stiefel-Whitney classes $sw_1 \in H^1(\Sigma, \mathbb{Z}/2)$ and $sw_2 \in H^2(\Sigma, \mathbb{Z}/2)$ of $\mathcal{V}$ define characteristic classes which help distinguish the connected components of maximal Higgs bundles. Thus, the space of maximal representations decomposes as

$$\mathrm{Rep}^{max}(\Gamma, \mathrm{SO}_0(2,n+1)) = \bigsqcup_{\substack{sw_1 \in H^1(\Sigma,\mathbb{Z}/2)\\ sw_2 \in H^2(\Sigma,\mathbb{Z}/2)}} \mathrm{Rep}^{max}_{sw_1,sw_2}(\Gamma, \mathrm{SO}_0(2,n+1))$$

where $\mathrm{Rep}^{max}_{sw_1,sw_2}(\Gamma, \mathrm{SO}_0(2,n+1))$ is the set of maximal representations such that the Stiefel-Whitney classes of the bundle $\mathcal{V}$ are sw_1 and sw_2.

When $n > 2$, these characteristic classes distinguish the connected components of maximal $\mathrm{SO}_0(2,n+1)$-Higgs bundles. In other words each of the sets $\mathrm{Rep}^{max}_{sw_1,sw_2}(\Gamma, \mathrm{SO}_0(2,n+1))$ is non-empty and connected [BGPG06]. Thus, for $n > 2$, the space $\mathrm{Rep}^{max}(\Gamma, \mathrm{SO}_0(2,n+1))$ has 2^{2g+1} connected components.

Proposition 2.31. *For $n > 2$ each connected component of maximal $\mathrm{SO}_0(2,n+1)$-representations contains a point in the Fuchsian locus from Definition 2.7.*

Proof. Let $\rho_{Fuch} : \Gamma \to \mathrm{SO}_0(2,1)$ be a Fuchsian representation and $\alpha : \Gamma \to \mathrm{O}(n)$ be an orthogonal representation. Consider the maximal $\mathrm{SO}_0(2,n+1)$-representation

$$\rho = \big(\rho_{Fuch} \otimes \det(\alpha)\big) \oplus \alpha$$

in the Fuchsian locus. The associated conformal Higgs bundle is given by

$$\begin{array}{ccc} \mathcal{IK} \xrightarrow{1} & \mathcal{I} & \xrightarrow{1} \mathcal{IK}^{-1} \\ & \oplus & \\ & \mathcal{V} & \end{array},$$

where $\mathcal{V}$ is the flat orthogonal bundle associated to the representation α. □

The case of maximal $\mathrm{SO}_0(2,3)$-representations is slightly different. Namely, when the first Stiefel-Whitney class of $\mathcal{V}$ vanishes, the structure group of $\mathcal{V}$ reduces to $\mathrm{SO}(2)$. In this case, $\mathcal{V}$ is isomorphic to $\mathcal{N} \oplus \mathcal{N}^{-1}$ for some line bundle $\mathcal{N}$ with non-negative degree. Furthermore, the holomorphic section β decomposes as $\beta = (\mu, \nu) \in H^0(\mathcal{N}^{-1}\mathcal{K}^2) \oplus H^0(\mathcal{N}\mathcal{K}^2)$. By stability, if $\deg(\mathcal{N}) \geq 0$, then $\mu \neq 0$. Thus, we have a bound $0 \leq \deg(\mathcal{N}) \leq 4g-4$.

The *Hitchin component* is the connected component of the representation variety $\mathrm{Rep}(\Gamma, \mathrm{SO}_0(2,3))$ containing the representations of the form $\iota_{irr} \circ \rho$ where $\rho : \Gamma \to \mathrm{SO}_0(2,1)$ is Fuchsian and $\iota_{irr} : \mathrm{SO}_0(2,1) \to \mathrm{SO}_0(2,3)$ is the unique (up to conjugation) irreducible representation. This component is maximal and corresponds to the case $\deg(\mathcal{N}) = 4g-4$, which implies $\mathcal{N} = \mathcal{K}^2$.

Proposition 2.32. [BGPG06] *The space of maximal $\mathrm{SO}_0(2,3)$-representations decomposes as*

$$\bigsqcup_{sw_1 \neq 0,\ sw_2} \mathrm{Rep}^{max}_{sw_1,sw_2}(\Gamma, \mathrm{SO}_0(2,3)) \ \sqcup \bigsqcup_{0 \leq d \leq 4g-4} \mathrm{Rep}^{max}_d(\Gamma, \mathrm{SO}_0(2,3)).$$

Here the Higgs bundles corresponding to representations in $\mathrm{Rep}^{max}_{sw_1,sw_2}(\Gamma,\mathrm{SO}_0(2,3))$ *are given by* (9) *with Stiefel-Whitney classes of* $\mathcal{V}$ *given by* sw_1 *and* sw_2 *and, for representations in* $\mathrm{Rep}^{max}_d(\Gamma,\mathrm{SO}_0(2,3))$, *the corresponding Higgs bundles have* $\mathcal{V}=\mathcal{N}\oplus\mathcal{N}^{-1}$ *with* $\deg(\mathcal{N})=d$. *Moreover, each of the above spaces is connected.*

Remark 2.33. The components $\mathrm{Rep}^{max}_d(\Gamma,\mathrm{SO}_0(2,3))$ are the $\mathrm{SO}_0(2,3)$-versions of maximal $\mathrm{Sp}(4,\mathbb{R})$-representations discovered by Gothen [Got01]. Hence, we will call the $4g-4$ components $\bigsqcup_{0<d\leq 4g-4}\mathrm{Rep}^{max}_d(\Gamma,\mathrm{SO}_0(2,3))$ *Gothen components.* In particular, Hitchin representations are Gothen representations corresponding to $d=4g-4$. The remaining components

$$\bigsqcup_{sw_1\neq 0,\ sw_2}\mathrm{Rep}^{max}_{sw_1,sw_2}(\Gamma,\mathrm{SO}_0(2,3))\ \sqcup\ \mathrm{Rep}^{max}_0(\Gamma,\mathrm{SO}_0(2,3))$$

will be called *reducible components*. The name is justified by Proposition 2.34.

The Gothen components and the reducible components have important differences. In particular, the Gothen components are smooth, and all representations in Gothen components that are not Hitchin representations are Zariski dense [BGPG12, Col16a]. While all reducible components contain representations in the Fuchsian locus. Thus we have:

Proposition 2.34. *The reducible components of maximal* $\mathrm{SO}_0(2,3)$*-representations are the components containing the Fuchsian representations as defined in Definition 2.7).*

3. Maximal space-like surfaces in $\mathbb{H}^{2,n}$

In this section, we look at the action of a maximal representation $\rho:\Gamma\to\mathrm{SO}_0(2,n+1)$ on the pseudo-Riemannian symmetric space $\mathbb{H}^{2,n}$. We show that this action preserves a unique maximal space-like surface, the Gauss map of which gives a minimal surface in the Riemannian symmetric space $\mathfrak{X}$ of $\mathrm{SO}_0(2,n+1)$. As a corollary, we prove Labourie's conjecture for maximal $\mathrm{SO}_0(2,n+1)$ representations (see Theorem 7 of the introduction).

3.1. The space $\mathbb{H}^{2,n}$.

In this section, we recall without proofs some classical facts about the pseudo-Riemannian symmetric spaces $\mathbb{H}^{2,n}$.

Definition 3.1. The space $\mathbb{H}^{2,n}\subset\mathbb{R}\mathbf{P}^{n+3}$ is the set of lines in $\mathbb{R}^{2,n+1}$ in restriction to which the quadratic form $\mathbf{q}$ is negative. The space $\widehat{\mathbb{H}}^{2,n}$ is the set of vectors u in $\mathbb{R}^{2,n+1}$ such that $\mathbf{q}(u)=-1$.

The natural projection from $\widehat{\mathbb{H}}^{2,n}$ to $\mathbb{H}^{2,n}$ is a covering of degree 2. The restriction of the quadratic form $\mathbf{q}$ induces a pseudo-Riemannian metric on $\mathbb{H}^{2,n}$ of signature $(2,n)$ and sectional curvature -1. The group $\mathrm{SO}_0(2,n+1)$ acts transitively on $\mathbb{H}^{2,n}$ preserving this pseudo-Riemannian metric.

Remark 3.2. The space $\mathbb{H}^{2,1}$ is a Lorentz manifold called the *anti-de Sitter space* of dimension 3. Some of the results presented in this section generalize known results for $\mathbb{H}^{2,1}$ (see [BS10]). Note however, that the Lie group $\mathrm{SO}_0(2,2)$ is isomorphic to a two-to-one cover of $\mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$, thus the case $n=1$ is quite special.

Compactification. The space $\mathbb{H}^{2,n}$ is compactified by the space of isotropic lines in $\mathbb{R}^{2,n+1}$:

Definition 3.3. The Einstein Universe $\mathbf{Ein}^{1,n} \subset \mathbb{R}\mathbf{P}^{n+3}$ is the set of isotropic lines in $\mathbb{R}^{2,n+1}$. The space $\widehat{\mathbf{Ein}}^{1,n}$ is the quotient of the space of isotropic vectors in $\mathbb{R}^{2,n+1}$ by the action of $\mathbb{R}_{>0}$ by homotheties.

The space $\mathbf{Ein}^{1,n}$ has a natural conformal class of pseudo-Riemannian metrics with signature $(1,n)$ which is invariant by the action of $\mathrm{SO}_0(2,n+1)$. It is thus the local model for conformally flat Lorentz manifolds.

Geodesics. The complete geodesics of $\mathbb{H}^{2,n}$ are the intersections of $\mathbb{H}^{2,n}$ with projective planes. These geodesics fall into three categories:

- *space-like geodesics* are intersections of $\mathbb{H}^{2,n}$ with projective planes of signature $(1,1)$,
- *light-like geodesics* are intersections of $\mathbb{H}^{2,n}$ with projective planes of (degenerate) signature $(0,1)$,
- *time-like geodesics* are intersections of $\mathbb{H}^{2,n}$ with projective planes of signature $(0,2)$.

Let u and v be two vectors in $\mathbb{R}^{2,n+1}$ such that $\mathbf{q}(u) = \mathbf{q}(v) = -1$ and $v \neq \pm u$. Then the projections $[u]$ and $[v]$ of u and v in $\mathbb{H}^{2,n}$ are joined by a unique geodesic, which is the intersection of $\mathbb{H}^{2,n}$ with the projective plane spanned by u and v. If this geodesic is space-like, then one can define the space-like distance $d_{\mathbb{H}^{2,n}}([u],[v])$ between $[u]$ and $[v]$ as the length of the geodesic segment joining them. Though this function is not an actual distance, it will be useful later on.

Proposition 3.4. *The points $[u]$ and $[v]$ are joined by a space-like geodesic if and only if $|\mathbf{q}(u,v)| > 1$. In that case, we have*

$$d_{\mathbb{H}^{2,n}}([u],[v]) = \cosh^{-1} |\mathbf{q}(u,v)| \ .$$

Warped product structure. It is sometimes very useful to picture $\widehat{\mathbb{H}}^{2,n}$ as a product of a $\mathbb{H}^2 \times \mathbb{S}^n$ endowed with a "twisted" metric. To do so, consider an orthogonal splitting $\mathbb{R}^{2,n+1} = \mathbb{R}^{2,0} \oplus \mathbb{R}^{0,n+1}$.

Proposition 3.5. *Let $\mathbb{D}$ be the disc of radius 1 in $\mathbb{R}^2$, and $\mathbb{S}^n$ the sphere of radius 1 in $\mathbb{R}^{n+1}$.*

- *The map*
$$\begin{array}{rccc} F: & \mathbb{D} \times \mathbb{S}^n & \to & \widehat{\mathbb{H}}^{2,n} \\ & (u,v) & \mapsto & \left(\frac{2}{1-\|u\|^2}u, \frac{1+\|u\|^2}{1-\|u\|^2}v\right) \end{array}$$
is a homeomorphism.
- *We have*
$$F^* g_{\mathbb{H}^{2,n}} = \frac{4}{(1-\|u\|^2)^2} g_{\mathbb{D}} \oplus - \left(\frac{1+\|u\|^2}{1-\|u\|^2}\right)^2 g_{\mathbb{S}^n} \ . \tag{10}$$
where $g_{\mathbb{D}}$ is the flat metric $dx^2 + dy^2$ and $g_{\mathbb{S}^n}$ is the spherical metric of curvature 1 on $\mathbb{S}^n$.
- *The map*
$$\begin{array}{rccc} \partial_\infty F: & \partial\mathbb{D} \times \mathbb{S}^n & \to & \widehat{\mathbf{Ein}}^{1,n} \\ & (u,v) & \mapsto & (u,v) \end{array}$$
is a homeomorphism that extends F continuously.

3.2. **Extremal surfaces.** Here we recall some basic facts about extremal immersions and refer to [Anc11, Chpater 1.3] for more details.

Consider a 2-dimensional oriented surface S and an n-dimensional manifold M endowed with a metric g of signature (p,q), with $p \geq 2$. An immersion $\iota : S \hookrightarrow M$ gives an identification of the tangent bundle TS to S with a sub-bundle of the pull-back bundle ι^*TM. This bundle is endowed with the pull-back metric ι^*g. The immersion ι is called *space-like* if the restriction of ι^*g to TS is positive definite. In that case one gets an orthogonal splitting

$$\iota^*TM = TS \oplus NS,$$

where NS is the orthogonal of TS with respect to ι^*g.

Let g_T and g_N denote the restrictions of ι^*g to TS and NS respectively and let ∇ be the pull-back of the Levi-Civita connection on M.

For X and Y vector fields on S and ξ a section of NS, the decomposition of ∇ along TS and NS gives

$$\begin{cases} \nabla_X Y &= \nabla^T_X Y + \mathrm{II}(X,Y) \\ \nabla_X \xi &= -B(X,\xi) + \nabla^N_X \xi \end{cases} .$$

Here, ∇^T is the Levi-Civita connection of (S, g_T), ∇^N is a metric connection on NS, $\mathrm{II} \in \Omega^1(S, \mathrm{Hom}(TS, NS))$ is called the *second fundamental form* and $B \in \Omega^1(S, \mathrm{Hom}(NS, TS))$ is called the *shape operator*.

Since ∇ is torsion-free, the second fundamental form is symmetric, namely, $\mathrm{II} \in \Omega^0(\mathrm{Sym}^2(TS)^* \otimes NS)$. Note also that

$$g_N\big(\mathrm{II}(X,Y),\xi\big) = g_T\big(B(X,\xi),Y\big).$$

The *mean curvature vector field* of the immersion $\iota : S \hookrightarrow M$ is given by

$$H := \mathrm{tr}_{g_T}(\mathrm{II}) \in \Omega^0(NS).$$

When S has co-dimension 1, the unit normal to the immersion allows II and H to be interpreted as real valued tensors. The following is classical:

Proposition 3.6. *The mean curvature field H vanishes identically if and only if the space-like immersion $\iota : S \hookrightarrow M$ is a critical point of the* area functional *which associates to ι the area of the metric g_T.*

We will call such an immersion an *extremal immersion*. When (NS, g_N) is positive definite, an extremal immersion locally minimizes the area will be called a *minimal immersion*. On the other hand, when (NS, g_N) is negative definite, an extremal immersion locally maximizes the area and will be called a *maximal immersion*.

Remark 3.7. When S is endowed with a conformal structure, ι is a space-like extremal immersion if and only if it is harmonic and conformal [ES64].

3.3. **Existence of maximal space-like surfaces.** In this Subsection, we prove the existence part of Theorem 8.

Proposition 3.8. *Let $\rho : \Gamma \longrightarrow \mathrm{SO}_0(2, n+1)$ be a maximal representation. Then there exists a ρ-equivariant maximal space-like immersion $u : \widetilde{\Sigma} \longrightarrow \mathbb{H}^{2,n}$.*

Proof. Let $X \in \mathcal{T}(\Sigma)$ be a critical point of the energy functional $\mathbf{E}_\rho : \mathcal{T}(\Sigma) \to \mathbb{R}_{>0}$, such an X exists by Proposition 2.14. By Theorem 2.30, the flat $\mathbb{R}^{2,n+1}$-bundle E_ρ with holonomy ρ decomposes orthogonally as

$$E_\rho = U \oplus \ell \oplus V \ ,$$

where ℓ is a negative definite line sub-bundle, U is positive definite of rank 2 and V is a negative definite of rank n. By pulling-back this splitting to $\widetilde{X}$, one sees that the negative definite line sub-bundle ℓ defines a ρ-equivariant map

$$u : \widetilde{X} \longrightarrow \mathbb{H}^{2,n}.$$

We will prove that u is a conformal harmonic immersion.

Over a local chart $U \subset \widetilde{X}$, the map u can be lifted to a map into $\widehat{\mathbb{H}}^{2,n} \subset \mathbb{R}^{2,n+1}$. The Levi-Civita connection of $\widehat{\mathbb{H}}^{2,n}$ is the connection induced by the flat connection ∇ on $\mathbb{R}^{2,n+1}$. Because $\widehat{\mathbb{H}}^{2,n}$ is umbilical, u satisfies the harmonic equation of Proposition 2.10 if and only if $\nabla_{\overline{\partial}_z}\nabla_{\partial_z} u$ is parallel to u (here, z is a complex coordinate on the local chart U).

Let

$$(\mathcal{E},\Phi) = \begin{array}{ccccc} \mathcal{I}\mathcal{K} & \xrightarrow{\ 1\ } & \mathcal{I} & \xrightarrow{\ 1\ } & \mathcal{I}\mathcal{K}^{-1} \\ & \nwarrow^{\beta^\dagger} & \oplus & \swarrow_{\beta} & \\ & & \mathcal{V} & & \end{array} \ ,$$

be the Higgs bundle associated to ρ as in Subsection 2.4 and let h be the Hermitian metric on $\mathcal{E}$ solving the Higgs bundle equations. The map u is locally given by a constant norm section of $\ell \subset \mathcal{I}$. Writing

$$\nabla = A + \Phi + \Phi^*,$$

where A is the Chern connection of $(\mathcal{E}, h)$, one gets

$$\begin{aligned} \nabla_{\overline{\partial}_z}\nabla_{\partial_z} u &= \left[\left((A^{0,1}+\Phi^*)(\overline{\partial}_z)\right)\circ (A^{1,0}+\Phi)(\partial_z)\right](u) \\ &= \left[\left((A^{0,1}+\Phi^*)(\overline{\partial}_z)\right)\circ \Phi(\partial_z)\right](u) \\ &= \left[\Phi^*\big(\overline{\partial}_z\big)\circ \Phi(\partial_z)\right](u). \end{aligned}$$

On the second line, we used the fact that the Chern connection is diagonal in the splitting and that u has constant norm, while for the third line, we used the holomorphicity of Φ.

In particular, $\Phi(\partial_z)u$ is a section of $\mathcal{L}^{-1}$. Since the splitting $\mathcal{E} = \mathcal{L}\oplus\mathcal{I}\oplus\mathcal{L}^{-1}\oplus\mathcal{V}$ is orthogonal with respect to the metric h, $\Phi^*\big(\overline{\partial}_z\big)$ sends $\mathcal{L}^{-1}$ on $\mathcal{I}$. Hence, $\nabla_{\overline{\partial}_z}\nabla_{\partial_z} u$ is a section of ℓ and is thus is parallel to u. Locally, the differential du corresponds to $\nabla u = (\Phi + \Phi^*)u = (1 + 1^*)u$ which is nowhere vanishing. In particular, u is an immersion.

The Hopf differential of u is locally given by

$$u^* q_{\mathbb{H}}^{2,0} = q_{\mathbb{H}}\big(\nabla_{\partial_z} u, \nabla_{\partial_z} u\big) dz^2,$$

where $q_{\mathbb{H}}$ is the $\mathbb{C}$-linear extension of the metric $g_{\mathbb{H}}$ on $\mathbb{H}^{2,n}$. But $\nabla_{\partial_z} u$ is a section of $\mathcal{L}^{-1}$ which is isotropic with respect to the $\mathbb{C}$-bilinear symmetric form q on $\mathcal{E}$. In particular, the Hopf differential is zero, which means that $u^* g_{\mathbb{H}}$ is a conformal metric on X. Since the map u is harmonic and conformal, it is a maximal immersion. ☐

Remark 3.9. In the splitting $E = U \oplus \ell \oplus V$, the bundle V is canonically identified with $NX := (u^* N\widetilde{X})/\rho(\Gamma)$, where $N\widetilde{X}$ is the normal bundle of the maximal space-like immersion $u : \widetilde{X} \longrightarrow \mathbb{H}^{2,n}$. In particular, the topology of the (quotient of the) normal bundle to $u : \widetilde{X} \longrightarrow \mathbb{H}^{2,n}$ characterizes the connected component of $\rho \in \mathrm{Rep}^{max}\big(\Gamma, \mathrm{SO}_0(2, n+1)\big)$.

Remark 3.10. The component of the Higgs field $\beta \in \Omega^{1,0}\big(X, \mathrm{Hom}(\mathcal{L}^{-1}, \mathcal{V})\big)$ is identified with the $(1,0)$-part of the second fundamental form $\mathrm{II} \in \Omega^1\big(X, \mathrm{Hom}(TX, NX)\big)$ of the maximal immersion u.

3.4. **Gauss maps.** Given a maximal representation $\rho \in \mathrm{Rep}^{max}\big(\Gamma, \mathrm{SO}_0(2, n+1)\big)$, let $u : \widetilde{X} \to \mathbb{H}^{2,n}$ be the ρ-equivariant maximal space-like immersion associated to a critical point $X \in \mathcal{T}(\Sigma)$ of the energy functional. In this Subsection, we describe different Gauss maps of the maximal surface u. In particular, we show that the ρ-equivariant minimal surface in the Riemannian symmetric space of $\mathrm{SO}_0(2, n+1)$ associated to the critical point X is a Gauss map of the maximal surface u.

Let us define the *main Grassmannian* $\mathbb{G}\big(\mathbb{R}^{2,n+1}\big)$ as the set of triple (F_0, F_1, F_2) where

- $F_0 \in \mathbb{H}^{2,n}$ is a negative definite line in $\mathbb{R}^{2,n+1}$,
- F_1 is a positive definite oriented 2-plane in $\mathbb{R}^{2,n+1}$ orthogonal to F_0,
- $F_2 = (F_0 \oplus F_1)^{\perp}$.

The stabilizer of a triple $(F_0, F_1, F_2) \in \mathbb{G}\big(\mathbb{R}^{2,n+1}\big)$ is the subgroup

$$\mathrm{H} := \mathrm{SO}(2) \times \mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(n)).$$

Hence, the main Grassmannian is the reductive homogeneous space

$$\mathbb{G}\big(\mathbb{R}^{2,n+1}\big) \cong \mathrm{SO}_0(2, n+1)/\mathrm{H}.$$

The map $p_1 : (F_0, F_1, F_2) \longmapsto F_0 \oplus F_1$ gives a projection of $\mathbb{G}\big(\mathbb{R}^{2,n+1}\big)$ the Grassmannian

$$\mathrm{Gr}_{(2,1)}\big(\mathbb{R}^{2,n+1}\big) = \mathrm{SO}(2, n+1)/\mathrm{S}(\mathrm{O}(2,1) \times \mathrm{O}(n))$$

of signature $(2,1)$ linear subspaces of $\mathbb{R}^{2,n+1}$.

Similarly, we have a projection $p_2 : (F_0, F_1, F_2) \longmapsto F_1$ of $\mathbb{G}\big(\mathbb{R}^{2,n+1}\big)$ onto the Grassmannian $\mathrm{Gr}_{(2,0)}\big(\mathbb{R}^{2,n+1}\big)$ of oriented space-like 2-planes in $\mathbb{R}^{2,n+1}$. Note that the Grassmannian

$$\mathrm{Gr}_{(2,0)}\big(\mathbb{R}^{2,n+1}\big) = \mathrm{SO}_0(2, n+1)/\mathrm{SO}(2) \times \mathrm{SO}(n+1)$$

is isomorphic to the Riemannian symmetric space $\mathfrak{X}$ of $\mathrm{SO}_0(2, n+1)$.

For $M, N \in \mathfrak{so}(2, n+1) \subset \mathfrak{sl}(n+3, \mathbb{R})$, the Killing form is given by

$$\langle M, N \rangle = (n+1)\mathrm{tr}(MN).$$

In particular, the Killing form is non-degenerate on the Lie algebra $\mathfrak{h}$ of H. Denote by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{h}$. The vector space decomposition $\mathfrak{h} \oplus \mathfrak{m}$ of $\mathfrak{so}(2, n+1)$ is Ad(H)-invariant. Hence, the Maurer-Cartan form of $\mathrm{SO}_0(2, n+1)$, $\omega \in \Omega^1\big(\mathrm{SO}_0(2, n+1), \mathfrak{so}(2, n+1)\big)$, decomposes as

$$\omega = \omega_{\mathfrak{h}} + \omega_{\mathfrak{m}},$$

where $\omega_{\mathfrak{h}} \in \Omega^1(\mathrm{SO}_0(2, n+1), \mathfrak{h})$ and $\omega_{\mathfrak{m}} \in \Omega^1(\mathrm{SO}_0(2, n+1), \mathfrak{m})$.

The H-equivariant form $\omega_{\mathfrak{m}}$ vanishes on vertical directions of the principal H-bundle $\mathrm{SO}_0(2, n+1) \longrightarrow \mathbb{G}(\mathbb{R}^{2,n+1})$, and so descends to $\omega_{\mathfrak{m}} \in \Omega^1\big(\mathbb{G}(\mathbb{R}^{2,n+1}), \mathrm{Ad}_H(\mathfrak{m})\big)$, where $\mathrm{Ad}_H(\mathfrak{m}) = \mathrm{SO}_0(2, n+1) \underset{\mathrm{Ad(H)}}{\times} \mathfrak{m}$ is the associated bundle with fiber $\mathfrak{m}$. For each point $x \in \mathbb{G}(\mathbb{R}^{2,n+1})$, the form $\omega_{\mathfrak{m}}$ gives an isomorphism $T_x\mathbb{G}(\mathbb{R}^{2,n+1}) \cong \mathfrak{m}$, and thus defines an identification

$$T\mathbb{G}(\mathbb{R}^{2,n+1}) \cong \mathrm{Ad}_H(\mathfrak{m}).$$

Finally, since the Killing form is Ad(H)-invariant and the splitting $\mathfrak{h} \oplus \mathfrak{m}$ is orthogonal, the Killing form defines a pseudo-Riemannian metric on $\mathbb{G}(\mathbb{R}^{2,n+1})$ of signature $(2n+2, n)$.

The same construction applies to the homogeneous spaces $\mathrm{Gr}_{(2,1)}\big(\mathbb{R}^{2,n+1}\big)$ and $\mathrm{Gr}_{(2,0)}\big(\mathbb{R}^{2,n+1}\big)$ where the Killing form induces a metric on signature $(2n, n)$ and $(2n+2, 0)$ respectively. The induced metric on $\mathrm{Gr}_{(2,0)}\big(\mathbb{R}^{2,n+1}\big)$ is the metric of the symmetric space of $\mathrm{SO}_0(2, n+1)$

Definition 3.11. Given a space-like immersion $v : S \longrightarrow \mathbb{H}^{2,n}$ of a surface S, the *main Gauss map of* u is the map

$$\mathcal{G} : S \longrightarrow \mathbb{G}\big(\mathbb{R}^{2,n+1}\big)$$

which sends a point $x \in S$ to the triple

$$(F_0(x), F_1(x), F_2(x)) := \big(v(x), dv(T_xS), (v(x) \oplus dv(T_xS))^{\perp}\big).$$

The *first* and *second Gauss map* are respectively defined to be $G_1 = p_1 \circ \mathcal{G}$ and $G_2 = p_2 \circ \mathcal{G}$.

We have the following:

Proposition 3.12. *The main, first and second Gauss maps of a ρ-equivariant maximal immersion $u : \widetilde{X} \longrightarrow \mathbb{H}^{2,n}$ are extremal space-like immersions.*

Proof. Since the calculations for each of the Gauss maps are similar, we will only prove the result for the main Gauss map. It is proved in [Ish82] that the three Gauss maps of a maximal immersion are harmonic. Thus, to prove the result we will show the Gauss maps are also conformal.

Recall that, given a signature $(2, n+1)$ scalar product $\mathbf{q}$ on $\mathbb{R}^{n+3}$, the Lie algebra of $\mathrm{SO}_0(2, n+1)$ is

$$\mathfrak{so}(2, n+1) = \big\{M \in \mathfrak{gl}_{n+3}(\mathbb{R}),\ \mathbf{q}M^T + M^T\mathbf{q} = 0\big\}.$$

Writing the matrices in blocks, with

$$\mathbf{q} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & -I_n \end{pmatrix},$$

where I_k is the identity matrix of size $k \times k$, we get that

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{pmatrix},\ A \in \mathfrak{so}(2),\ B \in \mathfrak{so}(n) \right\},$$

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & A & B \\ A^T & 0 & C \\ -B^T & C^T & 0 \end{pmatrix},\ A \in \mathcal{M}_{1,2}(\mathbb{R}),\ B \in \mathcal{M}_{1,n}(\mathbb{R}),\ C \in \mathcal{M}_{2,n}(\mathbb{R}) \right\}.$$

In particular, if $M = \begin{pmatrix} 0 & A & B \\ A^T & 0 & C \\ -B^T & C^T & 0 \end{pmatrix} \in \mathfrak{m}$, then

$$\langle M, M\rangle = 2(n+1)\left(AA^T - BB^T + \mathrm{tr}\left(CC^T\right)\right).$$

More explicitly, if $p = (F_0, F_1, F_2) \in \mathbb{G}(\mathbb{R}^{2,n+1})$, then we have an identification

$$T_p\mathbb{G}(\mathbb{R}^{2,n+1}) = \mathrm{Hom}(F_0, F_1) \oplus \mathrm{Hom}(F_0, F_2) \oplus \mathrm{Hom}(F_1, F_2),$$

and if $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in T_p\mathbb{G}(\mathbb{R}^{2,n+1})$, then the metric $g_{\mathbb{G}}$ induced by the Killing form is given by

$$g_{\mathbb{G}}(\varphi,\varphi) = 2(n+1)\left(\varphi_1\varphi_1^\dagger - \varphi_2\varphi_2^\dagger + \mathrm{tr}\left(\varphi_3\varphi_3^\dagger\right)\right),$$

where $\varphi_i^\dagger : F_i \longrightarrow F_{i-1}$ is obtained from $\varphi_i^* : F_i^* \longrightarrow F_{i-1}^*$ using the identification $F_i \cong F_i^*$ given by the induced scalar products.

Consider now $u : \widetilde{X} \longrightarrow \mathbb{H}^{2,n}$ a maximal immersion, and let $\mathcal{G} : \widetilde{X} \longrightarrow \mathbb{G}\big(\mathbb{R}^{2,n+1}\big)$ its associated main Gauss map. Given a point $x \in \widetilde{X}$, we get a canonical identification

$$T_{\mathcal{G}(x)}\mathbb{G} \cong \mathrm{Hom}\big(F_0(x), F_1(x)\big) \oplus \mathrm{Hom}\big(F_0(x), F_2(x)\big) \oplus \mathrm{Hom}\big(F_1(x), F_2(x)\big).$$

In particular, according to this splitting, we can write the differential as

$$d\mathcal{G} = \lambda + \mu + \nu.$$

Moreover, $\lambda \in \Omega^1\big(\widetilde{X}, \mathcal{G}^*(\mathrm{Hom}(F_0, F_1)\big)$ corresponds to the differential of u, $\mu \in \Omega^1\big(\widetilde{X}, \mathcal{G}^*(\mathrm{Hom}(F_0, F_2)\big)$ is zero by construction of the main Gauss map and $\nu \in \Omega^1\big(\widetilde{X}, \mathcal{G}^*(\mathrm{Hom}(F_1, F_2)\big)$ is identified with the second fundamental form of the immersion.

If $\partial\mathcal{G}$ denotes the $\mathbb{C}$-linear part of $d\mathcal{G}$ and by $q_{\mathbb{G}}$ the $\mathbb{C}$-linear extension of $g_{\mathbb{G}}$, then $\partial\mathcal{G} = \partial u + \beta$ where β is the $(1,0)$-part of the second fundamental form, and so is identified with the part of the Higgs field sending $\mathcal{L}^{-1}$ to $\mathcal{V}$ (see Remark 3.10).

The Hopf differential of $\mathcal{G}$ is thus given by

$$\begin{aligned} \mathrm{Hopf}(\mathcal{G}) &= q_{\mathbb{G}}(\partial\mathcal{G}, \partial\mathcal{G}) \\ &= 2(n+1)\mathrm{Hopf}(u) - 2(n+1)\mathrm{tr}\big(\beta\beta^\dagger\big) \\ &= -(n+1)\mathrm{tr}\big(\beta\beta^\dagger\big) \\ &= 0. \end{aligned}$$

For the last equation, we used the fact that $\beta^\dagger$ sends $\mathcal{V}$ to $\mathcal{L}$ (see subsection 2.4). Finally, a similar computation shows that $\mathcal{G}^*g_{\mathbb{G}} = (n+1)\|\Phi\|^2$, and thus never vanishes. In particular, $\mathcal{G}$ is a space-like immersion. □

3.5. **Uniqueness of the maximal surface.** Let $\rho \in \mathrm{Rep}^{max}(\Gamma, \mathrm{SO}_0(2, n+1))$ be a maximal representation. In this subsection, we prove the following theorem:

Theorem 3.13. *Let S_1 and S_2 be two connected ρ-invariant maximal space-like surfaces in $\mathbb{H}^{2,n}$ on which $\rho(\Gamma)$ acts co-compactly. Then $S_1 = S_2$.*

As a corollary, we prove Labourie's conjecture for maximal representations into Hermitian Lie groups of rank 2.

Corollary 3.14. *Let ρ be a maximal representation from Γ into a Hermitian Lie group of rank 2. Then the energy functional $\mathbf{E}_\rho: \mathcal{T}(\Sigma) \to \mathbb{R}$ defined in Subsection 2.3 has a unique critical point X. Moreover, the corresponding minimal immersion $f: \widetilde{X} \to \mathfrak{X}$ is an embedding.*

Note that when $n = 1$, Theorem 3.13 was obtained by Barbot, Béguin, Zeghib [BBZ03] and its corollary was obtained by Schoen [Sch93] (see Remark 3.27 for details).

Proof of Corollary 3.14 assuming Theorem 3.13 (as well as Corollary 3.21 and Corollary 3.14). By [BIW10], the Zariski closure of the image of $\rho(\Gamma)$ is of tube type; thus, we can assume that Γ takes values in $\mathrm{SO}_0(2, n+1)$ for some n (see Remark 1.10). Let X_1 and X_2 be two critical points of $\mathbf{E}_\rho$. Proposition 3.8 constructs two ρ-equivariant maximal space-like immersions $u_1 : \widetilde{X}_1 \to \mathbb{H}^{2,n}$ and $u_2 : \widetilde{X}_2 \to \mathbb{H}^{2,n}$. By Theorem 3.13, these two immersions have the same image S. Moreover, since S is homeomorphic to a disc (see Corollary 3.21), both u_1 and u_2 are diffeomorphisms onto S. The map $u_2 \circ u_1^{-1}$ induces a biholomorphism from X_1 to X_2 that is homotopic to the identity. Hence $X_1 = X_2$ in $\mathcal{T}(\Sigma)$.

Finally, by Proposition 3.12, the minimal ρ-equivariant immersion $f_1 : \widetilde{X} \to \mathfrak{X} = \mathrm{Gr}_{(2,0)}\left(\mathbb{R}^{2,n+1}\right)$ is the second Gauss map of the map u_1. Corollary 3.21 will show that u_1 is an embedding, and Corollary 3.17 will show that every negative definite linear subspace of $\mathbb{R}^{2,n+1}$ of dimension $n+1$ intersects $u_1(\widetilde{X}_1)$ exactly once. In particular, the second Gauss map of u_1 is injective, which concludes the proof of Corollary 3.14. □

In order to prove Theorem 3.13, we first need some elementary results about space-like surfaces in $\mathbb{H}^{2,n}$ invariant under the action of a maximal representation. Fix S a connected ρ-invariant space-like surface in $\mathbb{H}^{2,n}$ on which $\rho(\Gamma)$ acts co-compactly. We denote by $\partial_\infty S$ the topological boundary of S in the compactification $\mathbb{H}^{2,n} \cup \mathbf{Ein}^{1,n}$. Let $\widehat{S}$ denote the inverse image of S by the projection from $\widehat{\mathbb{H}}^{2,n}$ to $\mathbb{H}^{2,n}$.

Proposition 3.15. *The lift $\widehat{S}$ of S has at most two connected components diffeomorphic to discs. Moreover, if we identify $\widehat{\mathbb{H}}^{2,n}$ with $\mathbb{D} \times \mathbb{S}^n$ as in Proposition 3.5, then each of these connected components identifies with the graph of a Lipschitz map from $\mathbb{D}$ to $\mathbb{S}^n$.*

Remark 3.16. We will see in Corollary 3.21 that $\widehat{S}$ indeed has two connected components and that S itself is homeomorphic to a disc.

Proof. Denote the metric $\frac{4}{(1-\|u\|^2)^2} g_{\mathbb{D}}$ by $g_{\mathbb{H}^2}$, and let $\pi : \widehat{S} \to \mathbb{D}$ be the projection on the first factor. We have

$$\pi^* g_{\mathbb{H}^2} \geq g_{\mathbb{H}^{2,n}} ,$$

where $g_{\mathbb{H}^{2,n}}$ is the metric induced on $\widehat{S}$. Since $\widehat{S}$ is space-like and $\rho(\Gamma)$ acts co-compactly on $\widehat{S}$, the metric $g_{\mathbb{H}^{2,n}}$ is a complete Riemannian metric on $\widehat{S}$. Therefore, $\pi^* g_{\mathbb{H}^2}$ is also a complete Riemannian metric on $\widehat{S}$. It follows that $\pi : \widehat{S} \to \mathbb{H}^2$ is a proper immersion, hence a covering. Since $\mathbb{H}^2$ is simply connected and S is connected, $\widehat{S}$ has at most 2 connected components diffeomorphic to discs.

Let $\widehat{S}_0$ be one of the connected components of $\widehat{S}$. Since the projection $\widehat{S}_0$ to $\mathbb{D}$ is a diffeomorphism, $\widehat{S}$ is the graph of a $\mathcal{C}^1$ map $f : \mathbb{D} \to \mathbb{S}^n$. For every $z \in \mathbb{D}$ and every $v \in T_z\mathbb{D}$, we have

$$\frac{4}{\left(1-\|z\|^2\right)^2}\|v\|^2 - \left(\frac{1+\|z\|^2}{1-\|z\|^2}\right)^2 \|\mathrm{d}f_z(v)\|^2 > 0$$

since $\widehat{S}_0$ is space-like. Therefore,

$$\|\mathrm{d}f_z(v)\| < \frac{2}{1+\|z\|^2} \leq 2 \tag{11}$$

and f is Lipschitz. □

Note that one can choose the identification of $\widehat{\mathbb{H}}^{2,n}$ with $\mathbb{D}\times\mathbb{S}^n$ so that $\{0\}\times\mathbb{S}^n$ is the intersection of $\widehat{\mathbb{H}}^{2,n}$ with any given negative definite linear subspace of $\mathbb{R}^{2,n+1}$ of dimension $n+1$. One thus obtains the following corollary:

Corollary 3.17. *Any negative definite subspace of $\mathbb{R}^{2,n+1}$ of dimension $n+1$ intersects S exactly once.*

Let $\xi : \partial_\infty\Gamma \to \mathbf{Ein}^{1,n}$ be the ρ-equivariant boundary map from Theorem 2.4.

Lemma 3.18. *For every $\gamma\in\Gamma$, there exists a point $x\in S$ such that*

$$\rho(\gamma)^n\cdot x \underset{n\to+\infty}{\longrightarrow} \xi(\gamma_+)\ .$$

Proof. Fix $\gamma\in\Gamma$. By Corollary 2.5, one can find isotropic vectors e_+ and e_- in $\mathbb{R}^{2,n+1}$ with $\langle e_+,e_-\rangle = 1$ and a $\lambda>1$ such that $\rho(\gamma)\cdot e_+ = \lambda e_+$ and $\rho(\gamma)\cdot e_- = \frac{1}{\lambda}e_-$. Moreover, if V denotes the orthogonal of the vector space spanned by e_- and e_+, then the restriction of $\rho(\gamma)$ to V has spectral radius strictly less than λ.

Let x be a point in S and $\widehat{x}$ be a lift of x in $\widehat{\mathbb{H}}^{2,n}$. Up to scaling e_- and e_+, we can write

$$\widehat{x} = \alpha(e_- + e_+) + v\ ,$$

for some $\alpha\in\mathbb{R}$ and some $v\in V$. We thus have

$$\rho(\gamma)^n\cdot\widehat{x} = \lambda^n\alpha e_+ + \lambda^{-n}\alpha e_- + \rho(\gamma)^n v\ .$$

Since $\rho(\gamma)_{|V}$ has spectral radius strictly less than λ, we deduce that $\rho(\gamma)^n\cdot x$ converges (in $\mathbb{R}\mathbf{P}^{n+2}$) to $[e_+]=\xi(\gamma_+)$ unless $\alpha=0$.

Assume by contradiction that $\rho(\gamma)^n\cdot x$ does not converge to $\xi(\gamma_+)$ for any $x\in S$. In this case, S is included in $\mathrm{Proj}(V)$. However, this is not possible because the intersection of $\mathbb{H}^{2,n}$ with $\mathrm{Proj}(V)$ is a sub-manifold of signature $(1,n-1)$, and hence, cannot contain a space-like surface. □

Corollary 3.19. *The boundary of S in $\mathbb{H}^{2,n}\cup\mathbf{Ein}^{1,n}$ is the image of ξ. We denote it by $\partial_\infty S$.*

Proof. Let $\widehat{S}_0$ be a connected component of $\widehat{S}$. By Proposition 3.15, $\widehat{S}_0$ is the graph of a Lipschitz map $f:\mathbb{D}\to\mathbb{S}^n$. The map h extends to a continuous map $\partial f:\partial\mathbb{D}\to\mathbb{S}^n$ and the boundary of $\widehat{S}_0$ is the graph of ∂f (seen as a subset of $\widehat{\mathbf{Ein}}^{1,n}$). In particular, it is a topological circle, and so is its projection to $\mathbf{Ein}^{1,n}$.

Now, by Lemma 3.18, $\partial_\infty S$ contains $\xi(\gamma_+)$ for every $\gamma\in\Gamma$. Since the set $\{\gamma_+,\gamma\in\Gamma\}$ is dense in $\partial_\infty\Gamma$, we deduce that $\partial_\infty S$ contains the image of ξ. Since the image of ξ is also a topological circle, we conclude that $\partial_\infty S$ is exactly the image of ξ. □

Lemma 3.20. *Let x be a point in S. Then $S \cup \partial_\infty S$ does not intersect $x^\perp$.*

Proof. Let $\widehat{x}$ be a lift of x in $\widehat{\mathbb{H}}^{2,n}$ and $\widehat{S}_0$ the lift of S containing $\widehat{x}$. Since the space $\widehat{\mathbb{H}}^{2,n}$ is homogeneous, we can choose an identification of $\widehat{\mathbb{H}}^{2,n}$ with $\mathbb{D} \times \mathbb{S}^n$ so that $\widehat{x}$ is identified to the point $(0, v_0)$ for some $v_0 \in \mathbb{S}^n$.

Let $f : \overline{\mathbb{D}} \to \mathbb{S}^n$ be such that $\widehat{S}_0 \cup \partial_\infty \widehat{S}_0$ is the graph of f. In particular, we have $f(0) = v_0$. For z be a point in $\overline{\mathbb{D}}$, we have

$$\begin{aligned} d_{\mathbb{S}^n}(f(z), v_0) &\leq \int_0^{\|z\|} \left\| \frac{\mathrm{d}}{\mathrm{d}t} f(tz) \right\| \mathrm{d}t \\ &< \int_0^{\|z\|} \frac{2}{1+t^2} \text{ by (11)} \\ &< 2\arctan(\|z\|) \leq \frac{\pi}{2} \ . \end{aligned}$$

Since points orthogonal to $f(z)$ are at a distance $\frac{\pi}{2}$ in $\mathbb{S}^n$, v_0 is not orthogonal to $f(z)$, and we conclude that the point $(z, f(z)) \in \widehat{\mathbf{Ein}}^{1,n}$ is not in the orthogonal of $\widehat{x}$. Since this is true for any $z \in \overline{\mathbb{D}}$ and since $S \cup \partial_\infty S$ is the graph of f, this concludes the proof of the lemma. □

Corollary 3.21. *The lift of $S \cup \partial_\infty S$ to $\widehat{\mathbb{H}}^{2,n} \cup \widehat{\mathbf{Ein}}^{1,n}$ has two connected components, and S is homeomorphic to a disc.*

Proof. The projection from $\widehat{S} \cup \partial_\infty \widehat{S}$ to $S \cup \partial_\infty S$ is a covering of degree 2. Let x be a point in $\widehat{S}$. Then the function from $\partial_\infty \widehat{S}$ to $\{-1, 1\}$ associating to y the sign of $\langle x, y \rangle$ is a well-defined continuous function. Since $\langle x, -y \rangle = -\langle x, y \rangle$, this function takes both possible values and $\widehat{S} \cup \partial_\infty \widehat{S}$ thus has two connected components.

The covering of degree 2 from $\widehat{S}$ to S is thus a trivial covering. Since each connected component of $\widehat{S}$ is homeomorphic to a disc, so is S. □

Definition 3.22. Let $\partial_\infty \widehat{S}_0$ be one connected component of $\partial_\infty \widehat{S}$. The convex hull of $\partial_\infty \widehat{S}_0$ is the set of vectors $u \in \widehat{\mathbb{H}}^{2,n}$ such that any linear form on $\mathbb{R}^{2,n+1}$ which is positive on $\partial_\infty \widehat{S}_0$ is positive on u. The convex hull of $\partial_\infty S$, denoted $\mathrm{Conv}(\partial_\infty S)$, is the projection to $\mathbb{H}^{2,n}$ of the convex hull of either connected component of $\partial_\infty \widehat{S}$.

Proposition 3.23. *Assume that S is a maximal surface. Then S is included in the convex hull of $\partial_\infty S$.*

Proof. Let us choose $\widehat{S}_0$ a connected component of $\widehat{S}$, and let φ be a linear form $\mathbb{R}^{2,n+1}$ which is positive on $\partial_\infty \widehat{S}_0$. If u_0 is a point in $\widehat{S}_0$ and $\dot{u}_0$ a tangent vector to $\widehat{S}_0$ at u_0, then we have

$$\mathrm{Hess}_{u_0} \varphi_{|\widehat{S}_0}(\dot{u}_0) = q(\dot{u}_0)\varphi(u_0) + \varphi(\mathrm{II}(\dot{u}_0, \dot{u}_0)) \ ,$$

where II denotes the second fundamental form of $\widehat{S}_0$ in $\widehat{\mathbb{H}}^{2,n}$. Since $\widehat{S}_0$ is a maximal surface, the trace of II with respect to the metric induced by $\mathbf{q}$ on $\widehat{S}_0$ vanishes. We deduce that φ satisfies the equation

$$\Delta \varphi_{|\widehat{S}_0} = \varphi_{|\widehat{S}_0} \ ,$$

where Δ is the Laplace operator of the metric induced by $\mathbf{q}$ on $\widehat{S}_0$.

Now, by assumption, $\varphi_{|\widehat{S}_0}$ is positive in a neighborhood of $\partial_\infty \widehat{S}_0$. The classical maximum principle then implies that φ is positive on $\widehat{S}_0$. Therefore, $\widehat{S}_0$ is included in $\mathrm{Conv}\left(\partial_\infty \widehat{S}_0\right)$ and S is included in $\mathrm{Conv}(\partial_\infty S)$. □

We now turn to the proof of Theorem 3.13. Let S_1 and S_2 be two maximal ρ-invariant space-like surfaces in $\mathbb{H}^{2,n}$ on which ρ acts co-compactly. Assume by contradiction that S_1 and S_2 are distinct. Let us start by lifting S_1 and S_2 to $\widehat{\mathbb{H}}^{2,n}$ so that the two lifts have the same boundary. To simplify notations, we still denote those lifts by S_1 and S_2. Let $\langle \cdot, \cdot \rangle$ denote the symmetric bilinear form associated to the quadratic form $\mathbf{q}$ on $\mathbb{R}^{2,n+1}$.

Lemma 3.24. *For all $(u,v) \in S_1 \times S_2$,*

$$\langle u, v \rangle < 0 \ .$$

Proof. By Lemma 3.20, for any $u \in S_1$, the linear form $\langle u, \cdot \rangle$ is negative on $\partial_\infty S_1$. Moreover, since $\partial_\infty S_2 = \partial_\infty S_1$, Proposition 3.23 implies that S_2 is included in $\mathrm{Conv}\,(\partial_\infty S_1)$. Therefore, the linear form $\langle u, \cdot \rangle$ is negative on S_2. □

Lemma 3.25. *If $S_1 \neq S_2$, then there exists $(u,v) \in S_1 \times S_2$ such that*

$$\langle u, v \rangle > -1 \ .$$

Proof. Assume that S_1 is not included in S_2. Let x be a point in S_1 which is not in S_2. Choose identification of $\widehat{\mathbb{H}}^{2,n}$ with $\mathbb{D} \times \mathbb{S}^n$ for which x is identified to $(0, v_0)$ for some $v_1 \in \mathbb{S}^n$. Since S_2 is the graph of some function $f : \mathbb{D} \to \mathbb{S}^n$, there exists $v_2 \in \mathbb{S}^n$ such that $y = (0, v_2) \in S_2$. Since $x \notin S_2$, we have $v_2 \neq v_1$ and therefore

$$\langle x, y \rangle = -\langle v_1, v_2 \rangle > -1 \ .$$

□

Lemma 3.26. *The function*

$$\begin{array}{rccc} B: & S_1 \times S_2 & \to & \mathbb{R}_{>0} \\ & (u,v) & \mapsto & \langle u, v \rangle \end{array}$$

achieves its maximum.

Proof. Let $(u_n, v_n) \in (S_1 \times S_2)^{\mathbb{N}}$ be a maximizing sequence for B. Since $\rho(\Gamma)$ preserves B and acts co-compactly on S_1, we can assume that $(u_n)_{n\in\mathbb{N}}$ is bounded in S_1. Up to extracting a sub-sequence, we can assume that u_n converges to $u \in S_1$. By Lemma 3.25, we know that $B(u_n, v_n) > -1$ for n sufficiently large. Assume by contradiction that $(v_n)_{n\in\mathbb{N}}$ is unbounded in S_2. Up to extracting a sub-sequence, there exists $\varepsilon_n \underset{n\to+\infty}{\longrightarrow} 0$ such that $\varepsilon_n v_n$ converges to a vector $v \in \partial_\infty S_2$. Since $B(u_n, v_n)$ is bounded, we have

$$B(u,v) = \lim_{n\to+\infty} \varepsilon_n B(u_n, v_n) = 0 \ .$$

The vector v is thus in $u^\perp$. Since $\partial_\infty S_1 = \partial_\infty S_2$, this contradicts Lemma 3.20. □

We now have all the tools needed to apply the minimum principle to B and prove Theorem 3.13.

Proof of Theorem 3.13. Let $(u_0, v_0) \in S_1 \times S_2$ be a point where B achieves its maximum. By Lemmas 3.24 and 3.25, we have

$$-1 < B(u_0, v_0) < 0 \ .$$

For $\dot{u}_0 \in T_{u_0}S_1$ and $\dot{v}_0 \in T_{v_0}S_2$, let $(u(t))_{t\in(-\varepsilon,\varepsilon)}$ and $(v(t))_{t\in(-\varepsilon,\varepsilon)}$ be geodesic paths on S_1 and S_2 respectively, satisfying $u(0) = u_0$, $u'(0) = \dot{u}_0$ and $v(0) = v_0$, $v'(0) = \dot{v}_0$.

Since $B(u(t), v_0)$ is maximal at $t = 0$, we have $\langle \dot{u}_0, v_0\rangle = 0$. Since $\mathbf{q}(u(t)) = -1$ for all t, we also have $\langle \dot{u}_0, u_0\rangle = 0$. Similarly, we have $\langle \dot{v}_0, u_0\rangle = \langle \dot{v}_0, v_0\rangle = 0$. We thus obtain that $T_{u_0}S_1$ and $T_{v_0}S_2$ are both orthogonal to u_0 and v_0.

The second derivative of $B(u(t), v(t))$ at $t = 0$ is given by

$$\begin{aligned}(12) \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}_{|t=0} B(u(t), v(t)) = {} & 2\langle \dot{u}_0, \dot{v}_0\rangle + \langle \mathrm{II}_1(\dot{u}_0, \dot{u}_0), v_0\rangle + \langle \mathrm{II}_2(\dot{v}_0, \dot{v}_0), u_0\rangle \\ & + \mathbf{q}(\dot{u}_0)\langle u_0, v_0\rangle + \mathbf{q}(\dot{v}_0)\langle u_0, v_0\rangle \ ,\end{aligned}$$

where $\mathrm{II}_1 : T_{u_0}S_1 \times T_{u_0}S_1 \to u_0^\perp$ and $\mathrm{II}_2 : T_{v_0}S_2 \times T_{v_0}S_2 \to v_0^\perp$ denote respectively the second fundamental forms of S_1 and S_2 in $\widehat{\mathbb{H}}^{2,n}$. Our goal is to find $\dot{u}_0$ and $\dot{v}_0$ such that this second derivative is positive.

Since S_1 is a maximal surface in $\widehat{\mathbb{H}}^{2,n}$, the quadratic form $\beta_1 : w \mapsto \langle \mathrm{II}_1(w, w), v_0\rangle$ on $(T_{u_0}(S_1), \mathbf{q})$ has two opposite eigenvalues λ and $-\lambda$. Similarly, the quadratic form $w \mapsto \langle \mathrm{II}_2(w, w), v_0\rangle$ on $(T_{v_0}(S_2), \mathbf{q})$ has two opposite eigenvalues μ and $-\mu$. Up to switching S_1 and S_2, we may assume that $\lambda \geq \mu \geq 0$. We now choose $\dot{u}_0$ and $\dot{v}_0$ such that

$$\mathbf{q}(\dot{u}_0) = 1 \qquad \text{and} \qquad \dot{v}_0 = \frac{p(\dot{u}_0)}{\sqrt{\mathbf{q}(p(\dot{u}_0))}}$$

where $\beta_1(\dot{u}_0) = \lambda$ and $p : \{u_0, v_0\}^\perp \to T_{v_0}S_2$ denotes the orthogonal projection.

Since $\mathbf{q}(u_0) = \mathbf{q}(v_0) = -1$ and $|\langle u_0, v_0\rangle| < 1$, the restriction of $\mathbf{q}$ to $\mathrm{Vect}(u_0, v_0)$ is negative definite. The restriction of $\mathbf{q}$ to $\mathrm{Vect}(u_0, v_0)^\perp$ thus has signature $(2, n-2)$. Since $T_{v_0}S_2$ is a space-like plane in $\mathrm{Vect}(u_0, v_0)^\perp$, we can write $\dot{u}_0 = p(\dot{u}_0) + w$ where $\mathbf{q}(w) \leq 0$. We thus have

$$\mathbf{q}(p(\dot{u}_0)) = \mathbf{q}(\dot{u}_0) - \mathbf{q}(w) \geq \mathbf{q}(\dot{u}_0) = 1 \ ,$$

and therefore

$$\langle \dot{u}_0, \dot{v}_0\rangle = \sqrt{\mathbf{q}(p(\dot{u}_0))} \geq 1 \ .$$

Let us now get back to Equation (12). With our choices of $\dot{u}_0$ and $\dot{v}_0$, we have $\beta_1(\dot{u}_0) = \lambda$ and $\beta_2(\dot{v}_0) \geq -\mu \geq -\lambda$. Since $\langle u_0, v_0\rangle = B(u_0, v_0) > -1$, we have

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}_{|t=0} B(u(t), v(t)) &= 2\langle \dot{u}_0, \dot{v}_0\rangle + 2\langle u_0, v_0\rangle + \beta_1(\dot{u}_0) + \beta_2(\dot{v}_0) \\ &\geq 2\langle u_0, v_0\rangle + 2 \\ &> 0 \ .\end{aligned}$$

This contradicts the maximality of B at (u_0, v_0). $\square$

Remark 3.27 (Comparison with the work of Labourie and Bonsante–Schlenker)*.* In the case of $\mathrm{SO}_0(2, 2)$, Corollary 3.14 was proven directly by Schoen [Sch93] (see also Labourie [Lab92]). This case is quite special because $\mathrm{SO}_0(2, 2)$ is a degree 2 cover of $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{SO}_0(2, 2)/\mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(2))$ identifies with $\mathbb{H}^2 \times \mathbb{H}^2$.

Krasnov – Schlenker [KS07] and Bonsante – Schlenker [BS10] later clarified the link between maximal surfaces in $\mathbb{H}^{2,1}$ and minimal surfaces in $\mathbb{H}^2\times\mathbb{H}^2$. In [BS10], they gave an intrinsic proof of the uniqueness of a maximal surface in $\mathbb{H}^{2,1}$ in a more general setting. In their proof, they maximize the time-like distance between a point in S_1 and a point in S_2 and derive a contradiction from a maximum principle. This approach requires an estimate on the curvature of the maximal surface. Our strategy above is inspired by their proof, except that we apply the maximum principle to the scalar product instead of the space-like distance, which does not require any curvature estimate. This relieves us from extra technical difficulties.

3.6. **Length spectrum of maximal representations.** In this section, we exploit the pseudo-Riemannian geometry of $\mathbb{H}^{2,n}$ and the existence of a ρ-equivariant maximal space-like embedding of $\widetilde{\Sigma}$ to obtain a comparison of the length spectrum of ρ with that of a Fuchsian representation.

In our setting, we define the length spectrum of a representation ρ as follows.

Definition 3.28. Let ρ be a representation of Γ into $\mathrm{SO}_0(2,n+1)$. The *length spectrum* of ρ is the function $L_\rho:\Gamma\to\mathbb{R}_+$ that associates to an element $\gamma\in\Gamma$ the logarithm of the spectral radius of $L_\rho(\gamma)$ (seen as a squared matrix of size $n+3$).

Remark 3.29. Since, for $A\in\mathrm{SO}_0(2,n+1)$, A and A^{-1} have the same spectral radius, this definition coincides with Definition 1.3.

Theorem 3.30. *If $\rho:\Gamma\to\mathrm{SO}_0(2,n+1)$ is a maximal representation, then either ρ is in the Fuchsian locus (see Definition 2.7), or there exists a Fuchsian representation $j:\Gamma\to\mathrm{SO}_0(2,1)$ and $\lambda>1$ such that*

$$L_\rho \geq \lambda L_j\ .$$

Remark 3.31. The representation ρ is in the Fuchsian locus if and only if it stabilizes a totally geodesic space-like copy of $\mathbb{H}^2$ in $\mathbb{H}^{2,n}$. The induced action of ρ on $\mathbb{H}^2$ gives a Fuchsian representation j such that $L_j=L_\rho$.

Remark 3.32. Let m_{irr} denote the irreducible representation of $\mathrm{SO}_0(2,1)$ into $\mathrm{PSL}(n,\mathbb{R})$. For a Hitchin representation $\rho:\Gamma\to\mathrm{PSL}(n,\mathbb{R})$, one could hope to find a Fuchsian representation $j:\Gamma\to\mathrm{SO}_0(2,1)$ such that

$$L_\rho \geq L_{m_{irr}\circ j} = \frac{n-1}{2}L_j\ .$$

However, this statement fails to be true for $n\geq 4$ (see [LZ14, Section 3.3]). In particular, it is not true for Hitchin representations into $\mathrm{SO}_0(2,3)$ for which, nonetheless, Theorem 3.30 gives a weaker result.

In order to prove Theorem 3.30, let us fix a maximal representation $\rho:\Gamma\to\mathrm{SO}_0(2,n{+}1)$ and let $u:\widetilde{\Sigma}\to\mathbb{H}^{2,n}$ be a ρ-equivariant maximal space-like embedding. The pseudo-Riemannian metric on $\mathbb{H}^{2,n}$ induces a Riemannian metric g_u on Σ by restriction. By Poincaré's Uniformization Theorem, the metric g_u is conformal to a unique metric g_P of constant curvature -1.

Lemma 3.33. *Either ρ is in the Fuchsian locus, or there exists $\lambda>1$ such that $g_u\geq\lambda g_P$.*

Proof. Let $\kappa(g_u)$ denote the Gauss curvature of g_u. Recall that $\kappa(g_u)$ can be computed from the second fundamental form by the formula :

$$\kappa(g_u)_x = -1 - \sum_{i=1}^n \det_{g_u} \left\langle \mathrm{II}_{u(x)}(\cdot), e_i \right\rangle$$

where $(e_i)_{1\leq i\leq n}$ is an orthonormal basis of the orthogonal of $T_{f(x)}f(\Sigma)$. (Note that the minus sign in front of the sum comes from the fact that the metric of $\mathbb{H}^{2,n}$ is negative definite on this orthogonal.)

Since $f(\Sigma)$ is maximal, the quadratic form $\left\langle \mathrm{II}_{u(x)}(\cdot), e_i \right\rangle$ has trace 0 with respect to g_u and thus $\det_{g_u} \left\langle \mathrm{II}_{u(x)}, e_i \right\rangle \leq 0$, with equality if and only if $\mathrm{II}_{u(x)} = 0$. Therefore, $\kappa(g_u) \geq -1$, and if $\kappa(g_u) = -1$ everywhere, then $u(\widetilde{\Sigma})$ is totally geodesic. The Lemma now follows from the classical *Ahlfors–Schwarz–Pick lemma* (see for instance [Wol82]). □

Let g be a Riemannian metric on Σ and denote by d_g the associated distance on $\widetilde{\Sigma}$. We define the length spectrum of g as the map

$$\begin{array}{cccc} L_g : & \Gamma & \to & \mathbb{R}_+ \\ & \gamma & \mapsto & \lim_{n\to+\infty} \frac{1}{n}\, d_g(x, \gamma^n \cdot x) \end{array} ,$$

where x is any point in $\widetilde{\Sigma}$.

From now on, we assume that ρ does not preserve a copy of $\mathbb{H}^2$. It follows from Lemma 3.33 that $L_{g_u} \geq \lambda L_{g_P}$ for some $\lambda > 1$. Let j be the Fuchsian representation uniformizing g_P, i.e. such that there exists a j-equivariant isometry from $(\widetilde{\Sigma}, g_P)$ to $\mathbb{H}^2$. We then have

$$\lambda L_j = \lambda L_{g_P} \leq L_{g_u} \ .$$

In order to prove Theorem 3.30, it is thus enough to show the following:

Lemma 3.34. *We have*

$$L_\rho \geq L_{g_u} \ .$$

In order to prove this lemma, we need another characterization of L_ρ. Recall that $d_{\mathbb{H}^{2,n}}(x,y)$ denotes the length of the space-like geodesic segment between x and y. Recall that, if x and y are joined by a space-like geodesic, $d_{\mathbb{H}^{2,n}}(x,y)$ denotes the length of the space-like geodesic segment between x and y. We set $d_{\mathbb{H}^{2,n}}(x,y) = 0$ otherwise.

Proposition 3.35. *For any $\gamma \in \Gamma$ and any $x \in u(\widetilde{\Sigma})$, we have*

$$L_\rho(\gamma) = \lim_{n\to+\infty} \frac{1}{n} d_{\mathbb{H}^{2,n}}(x, \rho(\gamma)^n \cdot x) \ .$$

Proof. By Corollary 2.5, one can find two isotropic vectors e_+ and $e_- \in \mathbb{R}^{2,n+1}$ with $\langle e_+, e_-\rangle = 1$ such that $\rho(\gamma)\cdot e_+ = e^{L_\rho(\gamma)} e_+$ and $\rho(\gamma)\cdot e_- = e^{-L_\rho(\gamma)} e_-$. Moreover, if V denotes the orthogonal of the vector space spanned by e_- and e_+, then the spectral radius of the restriction of $\rho(\gamma)$ to V is strictly less than $e^L_\rho(\gamma)$.

Let $v \in \mathbb{R}^{2,n+1}$ be a vector of norm -1 whose projection $[v]$ to $\mathbb{H}^{2,n}$ lies in $u(\widetilde{\Sigma})$. We can write

$$v = \alpha_- e_- + \alpha_+ e_+ + w \ ,$$

with $w \in V$. By Proposition 3.19, we have

$$\rho(\gamma)^n \cdot [v] \underset{n\to+\infty}{\longrightarrow} [e_+]$$

and

$$\rho(\gamma)^n \cdot [v] \underset{n\to-\infty}{\longrightarrow} [e_-]$$

Hence α_+ and α_- are non-zero.

We have

$$\frac{1}{n} d_{\mathbb{H}^{2,n}}([v], \rho(\gamma)^n \cdot [v]) \ = \ \frac{1}{n} \cosh^{-1}\left(\langle v, \rho(\gamma)^n \cdot v\rangle\right) \ .$$

The right side of the equation is given by

$$\frac{1}{n}\cosh^{-1}\left|\left\langle \alpha_- e_- + \alpha_+ e_+ + w, \alpha_- e^{-nL_\rho(\gamma)} e_- + \alpha_+ e^{nL_\rho(\gamma)} e_+ + \rho(\gamma)^n \cdot w\right\rangle\right| \ ,$$

and thus,

$$\frac{1}{n} d_{\mathbb{H}^{2,n}}([v], \rho(\gamma)^n \cdot [v]) \ = \ \frac{1}{n}\cosh^{-1}\left|2\alpha_-\alpha_+ \cosh(nL_\rho(\gamma)) + \langle w, \rho(\gamma)^n \cdot w\rangle\right| \ .$$

Since the spectral radius of $\rho(\gamma)$ restricted to V is strictly less than $L_\rho(\gamma)$, the term $\langle w, \rho(\gamma)^n \cdot v\rangle$ is negligible and we obtain

$$\frac{1}{n} d_{\mathbb{H}^{2,n}}([v], \rho(\gamma)^n \cdot [v]) \underset{n\to+\infty}{\longrightarrow} L_\rho(\gamma).$$

□

In order to conclude the proof of Lemma 3.34, it suffices to prove the following:

Proposition 3.36. *If x and $y \in u(\widetilde{\Sigma})$ are joined by a space-like geodesic segment, then we have $d_u(x,y) \leq d_{\mathbb{H}^{2,n}}(x,y)$.*

Remark 3.37. Though we don't need it, one could easily deduce from the computations in the proof of Lemma 3.20 that two distinct points in $u(\widetilde{\Sigma})$ are *always* joined by a space-like geodesic.

Proof of Proposition 3.36. Recall that, according to Proposition 3.5, the space $\widehat{\mathbb{H}}^{2,n}$ is isometric to a warped product

$$\mathbb{H}^2 \times \mathbb{S}^n$$

with the metric

$$g = g_{\mathbb{H}^2} \oplus -w g_{\mathbb{S}^n} \ ,$$

for some positive function w on $\mathbb{H}^2$. In this warped product structure, the horizontal slices $\mathbb{H}^2 \times \{x_2\}$ are totally geodesic.

Let x and y be two points in $u(\widetilde{\Sigma})$ let $\widehat{x}$ and $\widehat{y}$ be lifts of x and y to $\widehat{\mathbb{H}}^{2,n}$ belonging to the same lift $\widehat{S}$ of $f(\widetilde{\Sigma})$. Let us choose a warped product structure on $\widehat{\mathbb{H}}^{2,n}$ such that x and y belong to the same horizontal slice.

Let π denote the restriction to $\widehat{S}$ of the projection on the $\mathbb{H}^2$ factor with respect to this warped product structure. We then have

$$d_{\mathbb{H}^{2,n}}(x,y) = d_{\mathbb{H}^2}(\pi(x), \pi(y)) \ .$$

By Proposition 3.15, π is a diffeomorphism. Moreover, given the warped product structure of the metric $g_{\mathbb{H}^{2,n}}$, we have

$$\pi^* g_{\mathbb{H}^2} \geq g_{\mathbb{H}^{2,n}} \tag{13}$$

on $u(\widetilde{\Sigma})$.

Let $c : [0,1] \to \mathbb{H}^2$ denote the geodesic segment between $\pi(x)$ and $\pi(y)$. We have

$$\begin{aligned} d_u(x,y) &\leq \int_0^1 \sqrt{g_{\mathbb{H}^2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\pi^{-1}\circ c(t)\right)}\mathrm{d}t \\ &\leq \int_0^1 \sqrt{g_{\mathbb{H}^2}\left(\frac{\mathrm{d}}{\mathrm{d}t}c(t)\right)}\mathrm{d}t = d_{\mathbb{H}^2}(\pi(x),\pi(y)) = d_{\mathbb{H}^{2,n}}(x,y)\ . \end{aligned}$$

□

We can now conclude that for any $\gamma \in \Gamma$,

$$\begin{aligned} L_{g_u}(\gamma) &= \lim_{n\to+\infty}\frac{1}{n}d_u(x,\gamma^n\cdot x) \\ &\leq \lim_{n\to+\infty}\frac{1}{n}d_{\mathbb{H}^{2,n}}(x,\gamma^n\cdot x) = L_\rho(\gamma)\ , \end{aligned}$$

which proves Lemma 3.34 and thus Theorem 3.30.

4. Geometric structures associated to maximal representations

In this section, we realize maximal representations in $\mathrm{SO}_0(2,n+1)$ as holonomies of geometric structures. More precisely, we prove the following two theorems:

Theorem 4.1. *The holonomy gives a surjective map from the space of fibered photon structures on iterated sphere bundles over Σ onto the set of maximal representations in $\mathrm{SO}_0(2,n+1)$.*

Theorem 4.2. *For any Hitchin representation $\rho \in \mathrm{Hit}(\Gamma, \mathrm{SO}_0(2,3))$, there exists a maximally fibered conformally flat Lorentz structure on the unit tangent bundle $\pi : T^1\Sigma \longrightarrow \Sigma$ whose holonomy is $\rho\circ\pi_*$.*

The notions of fibered photon structure, iterated sphere bundles, and maximally fibered conformally flat Lorentz structures are described in the next subsections.

4.1. (G,X)-structures.

Here we recall the basic theory of (G,X)-structures. For more details, the reader is referred to [Gol88a].

In this subsection, G will be a semi-simple Lie group, $X = G/H$ a G-homogeneous space and M a manifold such that $\dim(M) = \dim(X)$.

Definition 4.3. A (G,X)-structure on M is a maximal atlas of charts taking values in X whose transition functions are locally restriction of elements in G.

Two (G,X)-structures on M are *equivalent* if there exists a diffeomorphism $f : M \longrightarrow M$ isotopic to the identity whose expression in local charts is given by elements in G.

One can associate to a (G,X)-structure on M a *developing pair* (dev,ρ) where

$$\rho : \pi_1(M) \longrightarrow G$$

is called the *holonomy* of the structure and

$$\mathrm{dev} : \widetilde{M} \longrightarrow X$$

is a ρ-equivariant local homeomorphism called the *developing map*.

The developing pair is not uniquely defined. Given two developing pairs (dev_1, ρ_1) and (dev_2, ρ_2), if there exists an element $g \in G$ so that

$$\begin{cases} \mathrm{dev}_1 = g \circ \mathrm{dev}_2 \\ \rho_2(\gamma) = g \circ \rho_1(\gamma) \circ g^{-1}, \ \forall \gamma \in \pi_1(M) \end{cases},$$

then (dev_1, ρ_1) and (dev_2, ρ_2) correspond to equivalent (G, X)-structures. It is well-known (see for example [Gol88a]) that a developing pair fully determine the (G, X)-structure on M.

In particular, if $\mathcal{D}_{(G,X)}(M)$ is the moduli space of equivalence classes of (G, X)-structures on M, then we get a well-defined map

$$\mathbf{hol} : \mathcal{D}_{(G,X)}(M) \longrightarrow \mathrm{Rep}(\pi_1(M), G),$$

where $\mathrm{Rep}(\pi_1(M), G) := \mathrm{Hom}\big(\pi_1(M), G\big)/G$ is the representation variety.

The well-known *Ehresmann–Thurston principle* sates that this map induces a local homeomorphism from the set of equivalence classes of (G, X)-structures on M to the representation variety.

Theorem 4.4 (Thu80, Chapter 3)**.** *Let ρ_0 be the holonomy of a (G, X)-structure on a closed manifold M. Then any representation $\rho : \pi_1(M) \to G$ sufficiently close to ρ is the holonomy of a (G, X)-structure on M close to the initial one, which is unique up to equivalence.*

An X*-bundle over* M is a fiber bundle $p : \mathcal{X} \to M$ obtained by gluing together sets of the form $U_i \times X \cong p^{-1}(U_i)$, where $\{U_i\}_{i \in I}$ is a covering of M is an open set and for $U_i \cap U_j \neq \emptyset$, the transition functions have the form

$$\begin{array}{rlcl} \Psi : & (U_i \cap U_j) \times X & \longrightarrow & (U_i \cap U_j) \times X \\ & (m, x) & \longmapsto & \big(m, g(m)x\big) \end{array}$$

where g is a smooth map from $U_i \cap U_j$ to G.

Given a principal G-bundle $P \to M$, the quotient P/H is a X-bundle. Conversely, given an X-bundle over M, there exists an open covering $\mathcal{U} = \big\{U_i\big\}_{i \in I}$ of M such that the transition functions define a family of maps $g_{ij} : U_i \cap U_j \to G$ for any pair (i, j) with $U_i \cap U_j \neq \emptyset$. These maps satisfy the cocycle condition $g_{ij} g_{jk} g_{ki} = 1$ on triple intersections $U_i \cap U_j \cap U_k \neq \emptyset$. By gluing together sets of the form $U_i \times G$ with the same cocyle, we obtain a principal G-bundle that we call *the underlying principal G-bundle.*

Definition 4.5. Two principal G-bundles P_1 and P_2 over M are isomorphic if there exists a G-equivariant diffeomorphism from P_1 to P_2 lifting the identity on M(or, equivalently, if the principal G-bundle $\mathrm{Mor}(P_1, P_2)$ of morphisms from P_1 to P_2 admits a global section).

Two X-bundles are *isomorphic* if the underlying principal bundles are equivalent.

Given $\rho \in \mathrm{Rep}(\pi_1(M), G)$, one can associate an X-bundle $\mathcal{X}_\rho$ defined by

$$\mathcal{X}_\rho := P_\rho / H,$$

where P_ρ is the flat principal G-bundle with holonomy ρ. Equivalently, $\mathcal{X}_\rho = \big(\tilde{M} \times X\big)/\pi_1(M)$, where the action of $\gamma \in \pi_1(M)$ on $(m, x) \in \tilde{M} \times X$ is given by $\gamma.(m, x) = (\gamma.m, \rho(\gamma)x)$.

The bundle $\mathcal{X}_\rho$ is equipped with a flat structure, that is an integrable distribution of the dimension of M transverse to the fibers of $p : \mathcal{X}_\rho \longrightarrow M$. It follows that for each $x \in \mathcal{X}_\rho$, we have a splitting

$$T_x\mathcal{X}_\rho = T_x^v\mathcal{X}_\rho \oplus T_x^h\mathcal{X}_\rho.$$

Here $T_x^v\mathcal{X}_\rho = \ker(dp_x)$ is the vertical tangent space and $T_x^h\mathcal{X}_\rho$ is the horizontal tangent space given by the distribution. Note also that the projection $p : \mathcal{X}_\rho \longrightarrow M$ identifies $T_x^h\mathcal{X}_\rho$ with $T_{p(x)}M$.

In this language, a developing map wih holonomy ρ corresponds to a section s of $\mathcal{X}_\rho$ transverse to the horizontal distribution.

4.2. **Fibered photon structures.**

Definition 4.6. A *photon* in $\mathbb{R}^{2,n+1}$ is an isotropic 2-plane. We denote by $\mathbf{Pho}(\mathbb{R}^{2,n+1})$ the set of photons in $\mathbb{R}^{2,n+1}$.

Remark 4.7. Equivalently, a photon is a projective line inside the set of isotropic lines $\mathbf{Ein}^{1,n} \subset \mathbb{R}\mathbf{P}^{n+1}$. Such a projective line is necessarily the projectivization of an isotropic plane in $\mathbb{R}^{2,n+1}$.

The group $\mathrm{O}(2, n+1)$ acts transitively on $\mathbf{Pho}\big(\mathbb{R}^{2,n+1}\big)$ and the stabilizer of a photon is a parabolic subgroup denoted P. We thus get an identification

$$\mathbf{Pho}\big(\mathbb{R}^{2,n+1}\big) \cong \mathrm{O}(2, n+1)/P.$$

Lemma 4.8. *For $n > 0$, the space $\mathbf{Pho}(\mathbb{R}^{2,n+1})$ is diffeomorphic to the unit tangent bundle of the sphere $\mathbb{S}^n$ (sometimes called* iterated sphere*). In particular, $\mathbf{Pho}\left(\mathbb{R}^{2,2}\right) \cong \mathbb{S}^1 \sqcup \mathbb{S}^1$, $\mathbf{Pho}\left(\mathbb{R}^{2,3}\right) \cong \mathbb{RP}^3$ and for $n > 2$, $\mathbf{Pho}(\mathbb{R}^{2,n+1})$ is simply connected.*

Proof. Consider an orthogonal splitting $\mathbb{R}^{2,n+1} = E \oplus F$ where E is a positive definite 2-plane, $F = E^\perp$ and denote by g_E (respectively g_F) the scalar product induced on E (respectively F). For each photon $V \in \mathbf{Pho}(\mathbb{R}^{2,n+1})$, the orthogonal projection $p_E : \mathbb{R}^{2,n+1} \to E$ restricts to an isomorphism between V and E. In particular, each photon is the graph of a linear map $\varphi : E \to F$. We thus get an injective map

$$\Psi : \mathbf{Pho}(E \oplus F) \longrightarrow \mathrm{Hom}(E, F).$$

The image of Ψ consists of those linear maps $\varphi : E \to F$ such that $g_E(x, y) = -g_F(\varphi(x), \varphi(y))$ for any $x, y \in E$.

Fixing an orthonormal basis (e_1, e_2) of E, such a map $\varphi : E \to F$ is fully determined by the pair of orthonormal vectors $(\varphi(e_1), \varphi(e_2)) \in F^2$. The vector $\varphi(e_1)$ defines a point in $\mathbb{S}^n$ while $\varphi(e_2)$ is a unit vector orthogonal to $\varphi(e_1)$ and thus defines a point in $T^1_{\varphi(e_1)}\mathbb{S}^n$.

□

By considering $\mathbf{Pho}(\mathbb{R}^{2,n+1})$ as a sub-manifold of the Grassmannian of 2-planes in $\mathbb{R}^{n+3}$, one gets that $T_V\mathbf{Pho}(\mathbb{R}^{2,n+1}) \subset \mathrm{Hom}(V, \mathbb{R}^{2,n+1}/V)$. Given a negative definite line $\ell \in \mathbb{H}^{2,n}$, $\ell^\perp \cong \mathbb{R}^{2,n} \subset \mathbb{R}^{2,n+1}$ and one gets a natural embedding

$$\mathbf{Pho}\big(\ell^\perp\big) \hookrightarrow \mathbf{Pho}\big(\mathbb{R}^{2,n+1}\big).$$

The following lemma is straightforward (by a dimension argument):

Lemma 4.9. *For* $V \in \mathbf{Pho}(\ell^\perp) \subset \mathbf{Pho}\big(\mathbb{R}^{2,n+1}\big)$*, the post-composition with the orthogonal projection* $p_\ell : \mathbb{R}^{2,n+1} \to \ell$ *gives a linear morphism from* $T_V\mathbf{Pho}(\mathbb{R}^{2,n+1})$ *to* $\mathrm{Hom}(V,\ell)$ *the kernel of which is exactly* $T_V\mathbf{Pho}(\ell^\perp)$*.*

We now define a special class of photon structures:

Definition 4.10. A *fibered photon structure* is a $\big(\mathrm{O}(2,n{+}1),\mathbf{Pho}(\mathbb{R}^{2,n+1})\big)$-structure on a $\mathbf{Pho}\big(\mathbb{R}^{2,n}\big)$-bundle $\pi : M \to \Sigma$ whose holonomy along the fiber is trivial and whose developing map sends each fiber M_x bijectively to a copy of $\mathbf{Pho}\big(\mathbb{R}^{2,n}\big)$ in $\mathbf{Pho}\big(\mathbb{R}^{2,n+1}\big)$.

Two fibered photon structures on $\pi : M \to \Sigma$ are *equivalent* if there exists a diffeomorphism $f : M \to M$ isotopic to the identity giving an equivalence of $\big(\mathrm{O}(2,n+1),\mathbf{Pho}(\mathbb{R}^{2,n+1})\big)$-structure and preserving the fibers.

Remark 4.11. By definition, the holonomy representation of a fibered photon structure factors through a representation ρ of Γ into $\mathrm{O}(2,n{+}1)$ and its developing map descends to a ρ-equivariant local diffeomorphism

$$dev : \widetilde{\Sigma} \times \mathbf{Pho}\big(\mathbb{R}^{2,n}\big) \longrightarrow \mathbf{Pho}(\mathbb{R}^{2,n+1})$$

which sends the fibers bijectively onto copies of $\mathbf{Pho}(\mathbb{R}^{2,n})$ and which is equivariant with respect to $\rho : \Gamma \to \mathrm{O}(2,n+1)$. In particular, for $n = 2$, the image of dev has two connected components.

Given a fibered photon structure on M, *the associated surface* is the map

$$u : \begin{array}{ccc} \widetilde{\Sigma} & \longrightarrow & \mathbb{H}^{2,n} \\ x & \longmapsto & F_x^\perp, \end{array} \tag{14}$$

where $F_x \cong \mathbb{R}^{2,n}$ is such that $\mathrm{dev}(M_x) = \mathbf{Pho}(F_x)$. Such a map is equivariant with respect to the holonomy representation of the fibered photon structure.

Lemma 4.12. *Let* $\rho : \Gamma \to \mathrm{O}(2,n{+}1)$ *be the holonomy of a fibered photon structure. Then* ρ *is a maximal representation into the index two subgroup of* $\mathrm{O}(2,n+1)$*.*

Proof. By local injectivity, the developing map of a fibered photon structure sends the fibers of all points in a neighborhood of $x \in \widetilde{\Sigma}$ to disjoint photons in $\mathbf{Pho}\big(\mathbb{R}^{2,n+1}\big)$. For $y \in \widetilde{\Sigma}$ in a small neighborhood of x, $F_x \cap F_y = \big(u(x) \oplus u(y)\big)^\perp$. But $\mathrm{dev}(M_x) \cap \mathrm{dev}(M_y) = \emptyset$ if and only if $\big(u(x) \oplus u(y)\big)^\perp$ does not contain any isotropic plane, that is if and only if $\big(u(x) \oplus u(y)\big)^\perp \cong \mathbb{R}^{1,n}$. It follows that $u(x) \oplus u(y) \cong \mathbb{R}^{1,1}$, and so the geodesic passing through $u(x)$ and $u(y)$ is space-like. Since dev is a local diffeomorphism, an infinitesimal version of the above argument implies that the map u is smooth and space-like.

The second Gauss map of u (see Subsection 3.4) gives a reduction of structure group of the principal $\mathrm{O}(2,n+1)$-bundle P_ρ to a $\mathrm{O}(2) \times \mathrm{O}(n+1)$ principal bundle. Moreover, the underlying $\mathrm{O}(2)$ bundle is identified via u with the unit tangent bundle$T^1\Sigma$. In particular, it is orientable and the structure group thus reduces to $\mathrm{SO}(2) \times \mathrm{O}(n+1)$. Finally, it has Euler class $2-2g$. Therefore, the absolute value of the Toledo invariant of ρ is $2g-2$. ☐

4.3. **Constructing photon structures.** We will now show that, conversely, any maximal representation into $\mathrm{SO}_0(2,n{+}1)$ is the holonomy of a fibered photon structure. Note that, though we restricted ourselves to the connected component of the identity in $\mathrm{O}(2, n+1)$, one could easily extend the result to maximal representations into the index two subgroup of $\mathrm{O}(2, n+1)$ with maximal compact subgroup $\mathrm{SO}(2) \times \mathrm{O}(n+1)$.

Let $\rho \in \mathrm{Rep}^{max}\big(\Gamma, \mathrm{SO}_0(2, n+1)\big)$ be a maximal representation. By Theorem 2.30, the flat vector bundle E with holonomy ρ splits as

$$E = U \oplus \ell \oplus V,$$

where ℓ is a negative definite line sub-bundle, (U, g_U) is a positive definite rank 2 sub-bundle of Euler class $2g-2$ and (V, g_V) is a rank n negative definite sub-bundle. Recall also that the characteristic classes of V characterize the connected components of $\mathrm{Rep}^{max}\big(\Gamma, \mathrm{SO}_0(2, n+1)\big)$.

Set $\mathbf{Pho}(U \oplus V)$ the bundle over Σ whose fiber at $p \in \Sigma$ is the set $\mathbf{Pho}(U_p \oplus V_p)$ of photons in $U_p \oplus V_p$.

We have the following:

Lemma 4.13. *The canonical projection $\pi : \mathbf{Pho}(U \oplus V) \longrightarrow \Sigma$ turns $\mathbf{Pho}(U \oplus V)$ into a $\mathbf{Pho}(\mathbb{R}^{2,n})$-bundle. Moreover, the isomorphism class of the $\mathbf{Pho}(\mathbb{R}^{2,n})$-bundle is characterized by the degree of U and the topological type of V.*

Proof. The first point is obvious. To prove to second point, note that the principal $\mathrm{O}(2, n)$-bundle associated to the $\mathbf{Pho}(\mathbb{R}^{2,n})$-bundle $\pi : \mathbf{Pho}(U \oplus V) \to \Sigma$ corresponds to the reduction of structure group given by $\mathrm{U} \oplus V \subset E$. This bundle reduces to a $\mathrm{SO}(2) \times \mathrm{O}(n)$ bundle whose topological type is given by the degree of U (that is $2g-2$) and the topological type of V. □

We denote by $\mathbf{Pho}(E) = \big(\widetilde{\Sigma} \times \mathbf{Pho}(\mathbb{R}^{2,n+1})\big)/\Gamma$ the flat $\mathbf{Pho}(\mathbb{R}^{2,n+1})$-bundle over Σ associated to ρ. The fiber of $\mathbf{Pho}(E)$ over $p \in \Sigma$ is the set of photons in E_p. By Subsection 4.3, a photon structure on $\mathbf{Pho}(U \oplus V)$ with holonomy $\rho \circ \pi_*$ is given by a section $s \in \Omega^0\big(\mathbf{Pho}(U \oplus V), \pi^*\mathbf{Pho}(E)\big)$ which is transverse to the flat structure.

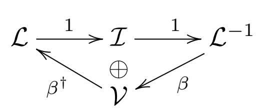
Using the canonical inclusion $U \oplus V \subset E$, one can define a tautological section $s \in \Omega^0\big(\mathbf{Pho}(U \oplus V), \pi^*\mathbf{Pho}(E)\big)$ of the bundle $\pi^*\mathbf{Pho}(E) \to \mathbf{Pho}(U \oplus V)$ by including a photon $\varphi \subset U_p \oplus V_p$ into E_p, for $p \in \Sigma$.

Proposition 4.14. *The section s introduced above defines a fibered photon structure on $\mathbf{Pho}(U \oplus V)$ of holonomy ρ.*

Proof. By construction, s maps bijectively the fiber of $\mathbf{Pho}(U \oplus V)$ over $p \in \Sigma$ to the set of photons in $U_p \oplus V_p = \ell_p^\perp$. In particular, for any $x = (p, \varphi) \in \mathbf{Pho}(U \oplus V)$, ds_x restricts to an isomorphism between $T_x^v\mathbf{Pho}(U \oplus V)$ and $T_{s(x)}\mathbf{Pho}(\ell_p^\perp) \subset T_{s(x)}\mathbf{Pho}(E_p)$.

The orthogonal projection on ℓ_p defines a map $P_{\ell_p} : T_{s(x)}\pi^*\mathbf{Pho}(E) \to \mathrm{Hom}(\varphi, \ell_p)$. By Lemma 4.9, s is transverse to the flat structure if the post-composition of the restriction of ds_x to $T_x^h\mathbf{Pho}(U \oplus V)$ with P_{ℓ_p} is injective.

From Subsection 2.4, the cyclic Higgs bundle associated to ρ splits as

$(\mathcal{E}, \Phi) \quad =$ 

where $\mathcal{I} = \Lambda^n \mathcal{V}$ is a square root of the trivial bundle and $\mathcal{L} = \mathcal{I}\mathcal{K}$. Moreover, this splitting is orthogonal with respect to the Hermitian metric h solving the Higgs bundle equations. In particular, for $\mathcal{E} = \mathcal{I} \oplus \mathcal{L} \oplus \mathcal{L}^{-1} \oplus \mathcal{V}$, we have

$$h = \begin{pmatrix} h_{\mathcal{I}} & & & \\ & h_{\mathcal{L}} & & \\ & & h_{\mathcal{L}}^{-1} & \\ & & & h_{\mathcal{V}} \end{pmatrix}$$

where $h_{\mathcal{I}}$ (respectively $h_{\mathcal{L}}$ and $h_{\mathcal{V}}$) is a Hermitian metric on $\mathcal{I}$ (respectively $\mathcal{L}$ and $\mathcal{V}$). The anti-linear involution fixing E_ρ preserves $\mathcal{I}$, $\mathcal{L} \oplus \mathcal{L}^{-1}$ and $\mathcal{V}$. The $(1,0)$ part $\nabla^{1,0}$ of the flat connection $\nabla = A + \Phi + \Phi^*$ (where A is the Chern connection of h and $\Phi^* = \Phi^{*h}$) is written

$$\nabla^{1,0} = \begin{pmatrix} A_{\mathcal{I}}^{1,0} & 1 & 0 & 0 \\ 0 & A_{\mathcal{L}}^{1,0} & 0 & \eta \\ 1 & 0 & A_{\mathcal{L}^{-1}}^{1,0} & 0 \\ 0 & 0 & \eta^\dagger & A_{\mathcal{V}}^{1,0} \end{pmatrix}$$

while the $(0,1)$-part $\nabla^{0,1}$ writes

$$\nabla^{0,1} = \begin{pmatrix} \overline{\partial}_{\mathcal{I}} & 0 & 1^* & 0 \\ 1^* & \overline{\partial}_{\mathcal{L}} & 0 & 0 \\ 0 & 0 & \overline{\partial}_{\mathcal{L}^{-1}} & (\eta^\dagger)^* \\ 0 & \eta^* & 0 & \overline{\partial}_{\mathcal{V}} \end{pmatrix}$$

where η^* is the $(0,1)$-form dual to $\eta \in \Omega^{0,1}\big(X, \mathrm{Hom}(\mathcal{L}^{-1}, \mathcal{V})\big)$, where the dual is taken using $h_{\mathcal{L}}^{-1}$ and h_V (and similarly for 1^* and $(\eta^\dagger)^*$).

Let ε be a local frame of $\mathcal{L}$ with $h_{\mathcal{L}}(\varepsilon, \varepsilon) = 1$, and $\lambda : \mathcal{E} \to \mathcal{E}$ be the anti-linear involution associated to h as in Theorem 2.18. The bundle $U = \mathrm{Fix}\big(\lambda_{|\mathcal{L}\oplus\mathcal{L}^{-1}}\big)$ is locally generated by the orthonormal frame $\frac{1}{\sqrt{2}}e_1, \frac{1}{\sqrt{2}}e_2$ where

$$\begin{cases} e_1 &= \varepsilon + \lambda(\varepsilon) \\ e_2 &= i(\varepsilon - \lambda(\varepsilon)) \end{cases}.$$

Identifying a photon $\psi \subset U \oplus V$ with the graph of a linear map $\varphi : U \to V$, the image of the section s is given by the sub-bundle generated by ξ_1 and ξ_2 where $\xi_i = e_i + \varphi(e_i) \in U \oplus V$.

We thus get

$$\xi_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ \varphi(e_1) \end{pmatrix}, \ \xi_2 = \begin{pmatrix} 0 \\ i \\ -i \\ \varphi(e_2) \end{pmatrix}.$$

In a local holomorphic coordinates z, denoting by $p_{\mathcal{I}} : \mathcal{E} \longrightarrow \mathcal{I}$ the orthogonal projection, we obtain

$$\begin{cases} p_{\mathcal{I}}(\nabla_{\partial_z}\xi_1) &= 1(\partial_z) \\ p_{\mathcal{I}}(\nabla_{\overline{\partial}_z}\xi_1) &= 1^*(\overline{\partial}_z) \\ p_{\mathcal{I}}(\nabla_{\partial_z}\xi_2) &= i1(\partial_z) \\ p_{\mathcal{I}}(\nabla_{\overline{\partial}_z}\xi_2) &= -i1^*(\overline{\partial}_z) \end{cases}.$$

where $1 \in \Omega^{1,0}\big(X, \mathrm{Hom}(\mathcal{I}, \mathcal{L})\big)$ and 1^* is the dual of 1 with respect to the Hermitian metrics on $\mathcal{I}$ and $\mathcal{L}$.

In particular, the post-composition of the restriction of ds to $T^v\mathbf{Pho}(U\oplus V)$ with the projection on ℓ is given by the matrix
$$\begin{pmatrix} 1(\partial_z) & i1(\partial_z) \\ 1^*(\overline{\partial}_z) & -i1^*(\overline{\partial}_z) \end{pmatrix}.$$
Since the determinant of this matrix is nowhere vanishing, s is transverse to the flat structure. Finally, note that the section s maps $\mathbf{Pho}(U\oplus V)_p$ to $\mathbf{Pho}(\ell_p^\perp)$, so the associated surface defined by equation 14 is the maximal one which is space-like. $\square$

The case $\mathrm{SO}_0(2,3)$. For the case of $\mathrm{SO}_0(2,3)$, one can say more about the topology of $\mathbf{Pho}(U\oplus V)$. Recall from Remark 2.33 that the space of maximal $\mathrm{SO}_0(2,3)$-representations decomposes as
$$\bigsqcup_{sw_1\neq 0,\ sw_2} \mathrm{Rep}^{max}_{sw_1,sw_2}(\Gamma,\mathrm{SO}_0(2,3)) \ \sqcup \bigsqcup_{0\leq d\leq 4g-4} \mathrm{Rep}^{max}_d(\Gamma,\mathrm{SO}_0(2,3)),$$
and that the components $\bigsqcup_{0<d\leq 4g-4} \mathrm{Rep}^{max}_d(\Gamma,\mathrm{SO}_0(2,3))$ are called Gothen components while the rest of the components are called reducible components.

By Theorem 2.30, a flat vector bundle E with holonomy representation $\rho\in\mathrm{Rep}^{max}(\Gamma,\mathrm{SO}_0(2,3))$ splits orthogonally as $E=U\oplus\ell\oplus V$ where ℓ is a negative definite line sub-bundle, U is an oriented positive definite rank two bundle canonically identified with $T\Sigma$ and $V=(U\oplus\ell)^\perp$. In this case, the post-composition by elements of $\mathrm{O}(2)$ turns the iterated sphere bundle $\mathbf{Pho}(U\oplus V)\to\Sigma$ into a principal $\mathrm{O}(2)$-bundle with the same first and second Stiefel-Whitney classes as V.

Lemma 4.15. *The total space of the $\mathbf{Pho}(\mathbb{R}^{2,2})$-bundle $\mathbf{Pho}(U\oplus V)\to\Sigma$ is connected if and only if the first Stiefel-Whitney class of V is non-zero.*

Proof. The splitting $U\oplus V$ gives a reduction of the principal $\mathrm{O}(2,2)$-bundle underlying $\mathbf{Pho}(U\oplus V)$ to a principal $\mathrm{SO}(2)\times\mathrm{O}(2)$-bundle. The stabilizer of a photon in $\mathbb{R}^{2,2}$ under the action of $\mathrm{SO}(2)\times\mathrm{O}(2)$ is conjugated to the diagonal embedding of $\mathrm{SO}(2)$, which is a connected subgroup. Therefore, $\mathbf{Pho}(U\oplus V)$ is connected if and only if the principal $\mathrm{SO}(2)\times\mathrm{O}(2)$-bundle is connected. This happens exactly when the first Stiefel-Whitney class of V is non-zero. $\square$

Recall that, for each maximal $\mathrm{SO}_0(2,3)$-representation, the associated fibered photon structure on $\mathbf{Pho}(U\oplus V)$ gives rise to a ρ-equivariant injective developing map $\mathrm{dev}:\widetilde{\Sigma}\times\mathbf{Pho}(\mathbb{R}^{2,2})\to\mathbf{Pho}(\mathbb{R}^{2,3})$ which sends each fiber bijectively onto a copy of $\mathbf{Pho}(\mathbb{R}^{2,2})$. In particular, the image of dev has two connected components. The geometry of the quotient $\rho(\Gamma)\backslash\mathrm{dev}(\widetilde{\Sigma}\times\mathbf{Pho}(\mathbb{R}^{2,2}))$ is given by the following:

Lemma 4.16. *Let ρ be a maximal $\mathrm{SO}_0(2,3)$-representation and $\mathbf{Pho}(U\oplus V)$ be the associated $\mathbf{Pho}(\mathbb{R}^{2,2})$-bundle.*

- *If ρ is in the Gothen component $\mathrm{Rep}^{max}_d(\Gamma,\mathrm{SO}_0(2,3))$, or in the reducible component $\mathrm{Rep}^{max}_0(\Gamma,\mathrm{SO}_0(2,3))$, then $\mathbf{Pho}(U\oplus V)$ is the disjoint union of two circle bundles with degrees $2g-2+d$ and $2g-2-d$.*
- *If ρ is in the reducible component $\mathrm{Rep}^{max}_{sw_1,sw_2}(\Gamma,\mathrm{SO}_0(2,3))$, then $\mathbf{Pho}(U\oplus V)$ is connected.*

Proof. In the first case, the first Stiefel–Whitney class of V vanishes and we can thus choose an orientation of V such that $\deg(V)=d\geq 0$. The two connected components of $\mathbf{Pho}(U\oplus V)$ are then given by the graphs of linear isometries φ :

$U \to V$ that preserve and reverse the orientation respectively. We respectively call them $\mathbf{Pho}^+(U \oplus V)$ and $\mathbf{Pho}^-(U \oplus V)$.

The complex structure $J_U : U \longrightarrow U$ given by the rotation of angle $\pi/2$ defines a canonical identification between U and $\mathrm{Ker}(J_U - i\mathrm{Id}) \cong \mathcal{K}^{-1} \subset U \otimes \mathbb{C}$. In a same way, the complex structure $J_V : V \longrightarrow V$ identifies V with a holomorphic line sub-bundle $\mathcal{N} \subset V \otimes \mathbb{C}$, and $\mathcal{N}$ has degree d. Under these identifications, $\mathbf{Pho}^+(U \oplus V)$ corresponds to unit vectors in $\mathrm{Hom}(\mathcal{K}^{-1}, \mathcal{N}) = \mathcal{K}\mathcal{N}$. Therefore, the degree of $\mathbf{Pho}^+(U \oplus V)$ is $2g - 2 + d$. In the same way, one gets that the degree of $\mathbf{Pho}^-(U \oplus V)$ is $2g - 2 - d$.

In the second case, the first Stiefel-Whitney class of V is non-zero, hence $\mathbf{Pho}(U \oplus V)$ is connected by Lemma 4.15. □

4.4. **Einstein structures for $\mathrm{SO}_0(2,3)$-Hitchin representations.** Here we prove Theorem 4.2, namely that one can to any $\mathrm{SO}_0(2,3)$-Hitchin representation a maximally fibered conformally flat Lorentz structure on the unit tangent bundle of Σ . More generally, we construct these structures for special $\mathrm{SO}_0(2,3)$ representations which give rise to cyclic Higgs bundles.

Definition 4.17. A *conformally flat Lorentz structure* (CFL structure) on a three dimensional manifold M is a (G,X)-structure with $G = \mathrm{SO}_0(2,3)$ and $X = \mathbf{Ein}^{1,2}$.

A *space-like circle* in $\mathbf{Ein}^{1,2}$ is the intersection of a 3-dimensional linear subspace of $\mathbb{R}^{2,3}$ of signature $(2,1)$ with $\mathbf{Ein}^{1,2}$. The set of space-like circles in $\mathbf{Ein}^{1,2}$ is the pseudo-Riemannian symmetric space

$$\mathrm{Gr}_{(2,1)}(\mathbb{R}^{2,3}) := \mathrm{SO}_0(2,3)/\mathrm{S}(\mathrm{O}(2,1) \times \mathrm{O}(2)).$$

Definition 4.18. A CFL structure on a circle bundle $\pi : M \longrightarrow \Sigma$ is called *fibered* if the developing map sends each fiber onto a space-like circle in $\mathbf{Ein}^{1,2}$ and the holonomy is trivial along the fiber.

Two fibered CFL structures on M are *equivalent* is there exists a diffeomorphism $f : M \to M$ isotopic to the identity giving an equivalence of $\big(\mathrm{SO}_0(2,3), \mathbf{Ein}^{1,2}\big)$-structures (see Definition 4.3) and moreover, f preserves the fibers.

In particular, the holonomy of a fibered space-like structure can thus be written as $\rho \circ \pi_*$ where $\rho : \Gamma \to \mathrm{SO}_0(2,3)$. Also, in a similar way to fibered photon structures, one can associate to a fibered CFL structure on M a ρ-equivariant map

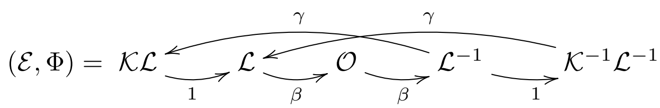
$$\Psi : \widetilde{\Sigma} \longrightarrow \mathrm{Gr}_{(2,1)}(\mathbb{R}^{2,3}).$$

The map Ψ sends a point $x \in \widetilde{\Sigma}$ to the element in $\mathrm{Gr}_{2,1}(\mathbb{R}^{2,3})$ corresponding to the space-like circle $\mathrm{dev}(\pi^{-1}(x))$.

Definition 4.19. A fibered CFL structure will be called *maximal* if $\Psi(\widetilde{\Sigma})$ is a space-like extremal surface.

Note that, up to the action of an element $g \in \mathrm{SO}_0(2,3)$, the surface $\Psi(\widetilde{\Sigma})$ only depends on the equivalence class of the fibered CFL structure.

Consider a representation $\rho \in \mathrm{Rep}\big(\Gamma, \mathrm{SO}_0(2,3)\big)$ such that there exists a Riemann surface structure $X \in \mathcal{T}(\Sigma)$ satisfying the property that the associated $\mathrm{SO}_0(2,3)$-Higgs bundle $(\mathcal{E}, \Phi)$ is cyclic and has the form



where $\mathcal{L}$ is a holomorphic line bundle of degree $0 \leq d \leq 2g-2$ and $\beta \in H^0(X, \mathcal{L}^{-1}\mathcal{K})$ is non-zero. In that case, the splitting $\mathcal{E} = \mathcal{KL} \oplus \mathcal{L} \oplus \mathcal{O} \oplus \mathcal{L}^{-1} \oplus \mathcal{K}^{-1}\mathcal{L}^{-1}$ is orthogonal with respect to the Hermitian metric h solving the Higgs bundle equations. Note that, Hitchin representations satisfy this property with $\mathcal{L} = \mathcal{K}$.

The associated anti-linear involution $\lambda : \mathcal{E} \longrightarrow \mathcal{E}$ fixing the flat $\mathrm{SO}_0(2,3)$-bundle E fixes $\mathcal{O}$, $\mathcal{L} \oplus \mathcal{L}^{-1}$ and $\mathcal{KL} \oplus \mathcal{K}^{-1}\mathcal{L}^{-1}$, and one gets a splitting

$$E = U \oplus \ell \oplus V,$$

where $\ell = \mathrm{Fix}(\lambda_{|\mathcal{O}})$ is trivial, $U = \mathrm{Fix}(\lambda_{|\mathcal{L}\oplus\mathcal{L}^{-1}})$ and $V = \mathrm{Fix}(\lambda_{|\mathcal{KL}\oplus\mathcal{K}^{-1}\mathcal{L}^{-1}})$.

Let $\pi : M \longrightarrow \Sigma$ be the circle bundle of Euler class d. Consider a tautological section $s_2 : M \longrightarrow \pi^*U$ normalized so that $\|s_2\|^2 = 1$ where the norm is taken with respect to the signature $(2,3)$ metric on π^*E. If s_1 is the section of the trivial line sub-bundle $\pi^*\ell$ normalized such that $\|s_1\|^2 = -1$, then the non-zero section $s = s_1 + s_2$ has zero norm. The section s thus defines a section σ of the flat homogeneous bundle $\pi^*\mathbf{Ein}(E)$ where

$$\mathbf{Ein}(E) := \big(P_\rho \times \mathbf{Ein}^{1,2}\big)/\mathrm{SO}_0(2,3)$$

and P_ρ is the flat $\mathrm{SO}_0(2,3)$-bundle with holonomy ρ. More concretely, the fiber of $\pi^*\mathbf{Ein}(E)$ over $x \in M$ is the set of isotropic vectors in $(\pi^*E)_x$.

Proposition 4.20. *The section $\sigma \in \Omega^0\big(M, \pi^*\mathbf{Ein}(E)\big)$ introduced above defines a maximally fibered CFL structure on M.*

Proof. In the splitting $\mathcal{E} = \mathcal{KL} \oplus \mathcal{L} \oplus \mathcal{O} \oplus \mathcal{L}^{-1} \oplus \mathcal{K}^{-1}\mathcal{L}^{-1}$, the Higgs field Φ and its dual Φ^* with respect to h have the following expression:

$$\Phi = \begin{pmatrix} 0 & 0 & 0 & \gamma & 0 \\ 1 & 0 & 0 & 0 & \gamma \\ 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \Phi^* = \begin{pmatrix} 0 & 1^* & 0 & 0 & 0 \\ 0 & 0 & \beta^* & 0 & 0 \\ 0 & 0 & 0 & \beta^* & 0 \\ \gamma^* & 0 & 0 & 0 & 1^* \\ 0 & \gamma^* & 0 & 0 & 0 \end{pmatrix},$$

where $\beta^* \in \Omega^{0,1}\big(X, \mathrm{Hom}(\mathcal{O}, \mathcal{L})\big) \cong \Omega^{0,1}\big(X, \mathrm{Hom}(\mathcal{L}^{-1}, \mathcal{O})\big)$ is the form dual to β using the Hermitian metric on $\mathcal{L}$ and $\mathcal{O}$ (and similarly for 1^* and γ^*).

Consider a local chart (z, θ) on $\widetilde{\Sigma} \times S^1$, where z is holomorphic. In this chart, the sections s_1 of $\pi^*\ell$ and s_2 of π^*U defined above write

$$s_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad s_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \mu^{-1}e^{i\theta} \\ 0 \\ \mu e^{-i\theta} \\ 0 \end{pmatrix},$$

where μ is the norm of the local section $\begin{pmatrix} 0 \\ e^{i\theta} \\ 0 \\ 0 \\ 0 \end{pmatrix}$ with respect to π^*h. In particular, if l is the local section of $\pi^*\mathcal{L}$ corresponding to $e^{i\theta}$, then the restriction of π^*h to $\pi^*(\mathcal{L}\oplus\mathcal{L}^{-1})$ is locally given by

$$\pi^*h_{|\pi^*(L\oplus L^{-1})} = \mu^2 l^{-1}\otimes \bar{l}^{-1} + \mu^{-2} l \otimes \bar{l}.$$

Writing the flat connection $\nabla = A + \Phi + \Phi^*$ (where $A = d + \partial \log h$ is the Chern connection of $(\pi^*\mathcal{E}, \pi^*h)$), one obtains

$$\nabla s_1 = \begin{pmatrix} 0 \\ \beta^*(s_1) \\ 0 \\ \beta(s_1) \\ 0 \end{pmatrix}.$$

The calculations for s_2 are more tedious. We get

$$A_{\partial_\theta} s_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ i\mu^{-1}e^{i\theta} \\ 0 \\ -i\mu e^{-i\theta} \\ 0 \end{pmatrix},\ A_{\partial_z} s_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \mu^{-2}\partial_z e^{i\theta}\mu \\ 0 \\ -\partial_z \mu e^{-i\theta} \\ 0 \end{pmatrix},\ A_{\overline{\partial}_z} s_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -\mu^{-2}\overline{\partial}_z \mu e^{i\theta} \\ 0 \\ \overline{\partial}_z \mu e^{-i\theta} \\ 0 \end{pmatrix},$$

and

$$\Phi(\partial_z)(s_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} \gamma(\partial_z)(\mu e^{-i\theta}) \\ 0 \\ \beta(\partial_z)(\mu^{-1}e^{i\theta}) \\ 0 \\ 1(\partial_z)(\mu e^{-i\theta}) \end{pmatrix},\ \Phi^*(\overline{\partial}_z)(s_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1^*(\overline{\partial}_z)(\mu e^{i\theta}) \\ 0 \\ \beta^*(\overline{\partial}_z)(\mu^{-1}e^{-i\theta}) \\ 0 \\ \gamma^*(\overline{\partial}_z)(\mu e^{i\theta}) \end{pmatrix}.$$

So finally, using $s = s_1 + s_2$, we get

$$\nabla_{\partial_z} s = \frac{1}{\sqrt{2}} \begin{pmatrix} \gamma(\partial_z)(\mu e^{-i\theta}) \\ \mu^{-2}\partial_z e^{i\theta}\mu \\ \beta(\partial_z)(\mu^{-1}e^{i\theta}) \\ -\partial_z \mu e^{-i\theta} + \sqrt{2}\beta(\partial_z)(s_1) \\ 1(\partial_z)(\mu e^{-i\theta}) \end{pmatrix},$$

$$\nabla_{\overline{\partial}_z} s = \frac{1}{\sqrt{2}} \begin{pmatrix} 1^*(\overline{\partial}_z)(\mu e^{i\theta}) \\ -\mu^{-2}\overline{\partial}_z \mu e^{i\theta} + \sqrt{2}\beta^*(\overline{\partial}_z)(s_1) \\ \beta^*(\overline{\partial}_z)(\mu^{-1}e^{-i\theta}) \\ \overline{\partial}_z \mu e^{-i\theta} \\ \gamma^*(\overline{\partial}_z)(\mu e^{i\theta}) \end{pmatrix}$$

and

$$\nabla_{\partial_\theta} s = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ i\mu^{-1}e^{i\theta} \\ 0 \\ -i\mu e^{-i\theta} \\ 0 \end{pmatrix}.$$

The section $\sigma \in \Omega^0\big(M, \pi^*\mathbf{Ein}(E)\big)$ is transverse to the flat structure if and only if the sections $\{s, \nabla_{\partial_z} s, \nabla_{\overline{\partial}_z} s, \nabla_{\partial_\theta} s\}$ generate a 4-dimensional space at each point. In particular the non-vanishing of the determinant $\big|s_1\ s_2\ \nabla_{\partial_z} s\ \nabla_{\overline{\partial}_z} s\ \nabla_{\partial_\theta} s\big|$ is a sufficient condition.

The three vectors $\{s_1, s_2, \nabla_{\partial_\theta} s\}$ span the bundle $\pi^*(\mathcal{L} \oplus \mathcal{O} \oplus \mathcal{L}^{-1})$ at each point. In particular, the determinant $\big|s_1\ s_2\ \nabla_{\partial_z} s\ \nabla_{\overline{\partial}_z} s\ \nabla_{\partial_\theta} s\big|$ vanishes exactly when the first and last component of $\{\nabla_{\partial_z} s,\ \nabla_{\overline{\partial}_z} s\}$ are proportional, that is when $\|\gamma\|^2 = \|1\|^2$.

Because the section $\gamma \in H^0(X, \mathcal{K}^2\mathcal{L}^2)$ is holomorphic, we have

$$\Delta \log \|\gamma\|^2 = -2F_{\mathcal{KL}},$$

where $F_{\mathcal{KL}}$ is the curvature of the bundle $\mathcal{KL}$ with respect to the Hermitian metric h. By the Higgs bundle equation, $F_{\mathcal{KL}} = \|1\|^2 - \|\gamma\|^2$ and we obtain

$$\Delta \log \|\gamma\|^2 = 2\|\gamma\|^2 - 2\|1\|^2.$$

The maximum principle applies: at a maximum of $\|\gamma\|^2$, one has $\|\gamma\|^2 < \|1\|^2$ and so $\|1\|^2 \neq \|\gamma\|^2$ on Σ. In particular, $\sigma \in \Omega^0\big(M, \pi^*\mathbf{Ein}(E)\big)$ defines a CFL structure on M.

Note also that the associated developing map sends the fiber of M over x to the space-like circle corresponding to the signature $(2,1)$ linear subspace $\ell_x \oplus U_x$, so the CFL structure is fibered. Finally, the corresponding equivariant map $\Psi : \widetilde{\Sigma} \longrightarrow \mathrm{Gr}_{2,1}(\mathbb{R}^{2,3})$ is the first Gauss map of the maximal surface $u : \widetilde{\Sigma} \longrightarrow \mathbb{H}^{2,2}$. By Proposition 3.12, Ψ is extremal. □

Remark 4.21. For $\mathcal{L} = \mathcal{K}$, the above construction gives maximally fibered CFL structures on $T^1\Sigma$ whose holonomy factors through a Hitchin representation. But note also that, for any $d \in \mathbb{Z}$ with $|d| < 2g - 2$, our construction gives examples of maximally fibered CFL structures on a degree d circle bundle over Σ whose holonomy factor through representations in the connected component of $\mathrm{Rep}(\Gamma, \mathrm{SO}_0(2,3))$ of Toledo invariant d. Unfortunately, for $|d| < 2g - 2$, these representations do not form an open domain of the character variety and we do not know how to characterize the representations arising this way. One can show that these representations do not come from representations in $\mathrm{SO}(2,2)$, so these CFL structures do not come from AdS structures on the circle bundle. It would be interesting to understand whether these representations are Anosov and whether the Einstein structures constructed above are deformation of anti-de Sitter structures.

5. Relation with Guichard-Wienhard construction

In this section, we show that both the fibered photon structure of Theorem 4.1 and the maximal CFL structures of Theorem 4.2 agree with the geometric structures constructed by Guichard-Wienhard in [GW12]. As a corollary, we describe the topology of the geometric structures of Guichard-Wienhard.

5.1. **Geometric structures "à la Guichard-Wienhard".** Here we explain the construction of geometric structures in [GW12] in the case of Anosov representations of a surface group in $\mathrm{SO}_0(2,n+1)$. Let P_1 and P_2 be respectively the stabilizer of an isotropic line and of an isotropic 2-plane in $\mathbb{R}^{2,n+1}$. In particular, $\mathrm{SO}_0(2,n+1)/P_1 \cong \mathbf{Ein}^{1,n}$ is the Einstein Universe and $\mathrm{SO}_0(2,n+1)/P_2 \cong \mathbf{Pho}(\mathbb{R}^{2,n+1})$ is the set of photons in $\mathbb{R}^{2,n+1}$.

Given $\rho \in \mathrm{Rep}(\Gamma, \mathrm{SO}_0(2,n+1))$ a representation which is P_i-Anosov ($i=1,2$), there exists a continuous ρ-equivariant map

$$\xi_i : \partial_\infty \Gamma \longrightarrow \mathrm{SO}_0(2,n+1)/P_i.$$

The following was established in [Lab06] for Hitchin representations, in [BILW05] for maximal representations in $\mathrm{Sp}(2n,\mathbb{R})$, so using the existence of a tight embedding $\iota : \mathrm{SO}_0(2,n+1) \hookrightarrow \mathrm{Sp}(2m,\mathbb{R})$ for some $m \in \mathbb{N}$ (see [HP14]), we get:

Proposition 5.1. *If $\rho \in \mathrm{Rep}(\Gamma, \mathrm{SO}_0(2,n+1))$ is a maximal representation then it is P_1-Anosov. If $\rho \in \mathrm{Rep}(\Gamma, \mathrm{SO}_0(2,3))$ is a Hitchin representation, then ρ is both P_1-Anosov and P_2-Anosov.*

If ρ is P_1-Anosov, define the subsets $K_\rho^2 \subset \mathbf{Pho}(\mathbb{R}^{2,n+1})$ by

$$K_\rho^2 := \left\{ V \in \mathbf{Pho}(\mathbb{R}^{2,n+1}) \mid \xi_1(x) \subset V \text{ for some } x \in \partial_\infty \Gamma \right\},$$

if ρ is P_2-Anosov define the subsets $K_\rho^1 \subset \mathbf{Ein}^{1,n}$ by

$$K_\rho^1 := \left\{ \ell \in \mathbf{Ein}^{1,n} \mid \ell \subset \xi_2(x) \text{ for some } x \in \partial_\infty \Gamma \right\}.$$

Note that K_ρ^2 is homeomorphic to $\partial_\infty\Gamma \times \mathbb{S}^{n-1}$. If $\rho \in \mathrm{Rep}(\Gamma, \mathrm{SO}_0(2,3))$ a P_2-Anosov representation, K_ρ^1 is homeomorphic to $\partial_\infty \Gamma \times \mathbb{S}^1$. The following is proved in [GW12]:

Theorem 5.2. *If $\rho \in \mathrm{Rep}\big(\Gamma, \mathrm{SO}_0(2,n+1)\big)$ is P_1-Anosov, then $\rho(\Gamma)$ acts properly discontinuously and co-compactly on the set*

$$\Omega_\rho^2 = \mathbf{Pho}(\mathbb{R}^{2,n+1}) \setminus K_\rho^2.$$

Also, if $\rho \in \mathrm{Rep}\big(\Gamma, \mathrm{SO}_0(2,n+1)\big)$ is P_2-Anosov, then $\rho(\Gamma)$ acts properly discontinuously and co-compactly on the set

$$\Omega_\rho^1 = \mathbf{Ein}^{1,n} \setminus K_\rho^1.$$

Moreover, the topology of the quotient $\rho(\Gamma)\backslash\Omega_\rho^i$ remains constant as the representation ρ is varied continuously (Theorem 9.2 of [GW12]).

5.2. **Equivalence of the photon structures.** Here we prove that the fibered photon structures constructed in Theorem 4.1 are equivalent to those of Guichard-Wienhard.

Theorem 5.3. *Let ρ be a maximal representation from Γ to $\mathrm{SO}_0(2,n+1)$. Let $\mathbf{Pho}(U \oplus V)$ be the iterated sphere bundle of Section 4.3 and dev the developing map of the photon structure on $\mathbf{Pho}(U \oplus V)$ constructed in Proposition 4.14. Then dev takes values in Ω_ρ and induces a diffeomorphism from $\mathbf{Pho}(U \oplus V)$ to $\rho(\Gamma)\backslash\Omega_\rho^2$*

Proof. Let $\rho \in \mathrm{Rep}^{max}(\Gamma, \mathrm{SO}_0(2,n+1))$ be a maximal representation, denote by $u : \widetilde{\Sigma} \to \mathbb{H}^{2,n}$ the ρ-equivariant maximal surface and by $\xi : \partial_\infty\Gamma \to \mathbf{Ein}^{1,n} \cong \partial\mathbb{H}^{2,n}$ the ρ-equivariant continuous map given by the Anosov property of ρ. Recall that the boundary of $u(\widetilde{\Sigma})$ corresponds to $\xi(\partial_\infty\Gamma)$. We will show that the developing map of the fibered photon structure of Theorem 4.1 maps bijectively onto the Guichard-Wienhard domain Ω^2_ρ.

In the construction of the fibered photon structure of Theorem 4.1, the developing map sends the fiber of the iterated sphere bundle over a point $x \in \widetilde{\Sigma}$ bijectively to the set of photons contained in the orthogonal of $u(x)$ in $\mathbb{R}^{2,n+1}$. By Lemma 3.20, the boundary of $u(\widetilde{\Sigma})$ does not intersect $u(x)^\perp$ for any $x \in \widetilde{\Sigma}$. In particular, the developing map of the space-like fibered photon structure associated to ρ is contained in the domain Ω^2_ρ.

For the other inclusion, suppose $V \in \mathbf{Pho}(\mathbb{R}^{2,n+1})$ is a photon and denote its orthogonal by $V^\perp$. The restriction of the quadratic form $\mathbf{q}$ to $V^\perp$ is non-positive, and vanishes exactly on the subspace V. Thus, the subspace $V^\perp$ can be approximated by a sequence W_k of rank $(n+1)$ negative definite subspaces. By Corollary 3.17, each plane W_k intersects the surface $u(\widetilde{\Sigma})$ in exactly one point. Thus, $V^\perp$ intersects either $u(\widetilde{\Sigma})$ or its boundary. This gives rise to a dichotomy:

- If $V^\perp$ intersects $u(\widetilde{\Sigma})$ at a point x, then V is contained in $\mathbf{Pho}(x^\perp)$ and in the image of developing map of the fibered photon structure.
- If $V^\perp$ intersects the boundary of $u(\widetilde{\Sigma})$ at a point $\xi(x)$, then V contains the isotropic line $\xi(x)$, and so V belongs to K^2_ρ.

Therefore, the developing map of the fibered photon structure from Theorem 4.1 maps surjectively onto Ω^2_ρ. □

The following corollary is immediate:

Corollary 5.4. *If $\rho : \Gamma \longrightarrow \mathrm{SO}_0(2, n+1)$ is a maximal representation, then the quotient $\rho(\Gamma)\backslash\Omega^2_\rho$ of the Guichard-Wienhard discontinuity domain is homeomorphic to an iterated sphere bundle over Σ and the topology of the bundle characterizes the connected component of ρ.*

By Lemma 4.16, for $\mathrm{SO}_0(2, 3)$ we can say a little more.

Corollary 5.5. *For $\rho : \Gamma \longrightarrow \mathrm{SO}_0(2, 3)$ maximal, the quotient $\rho(\Gamma)\backslash\Omega^2_\rho$ of the Guichard-Wienhard discontinuity domain*

- *is homeomorphic to a connected $\mathrm{O}(2)$-bundle over Σ with Steifel-Whitney classes (sw_1, sw_2) if $\rho \in \mathrm{Rep}^{max}_{sw_1,sw_2}(\Gamma, \mathrm{SO}_0(2, 3))$*
- *is homeomorphic to the disjoint union of two circle bundles of degree $2g - 2 + d$ and $2g - 2 - d$ if $\rho \in \mathrm{Rep}^{max}_d(\Gamma, \mathrm{SO}_0(2, 3))$.*

The Hitchin component. For a representation $\rho \in \mathrm{Hit}\big(\Gamma, \mathrm{SO}_0(2, 3)\big)$ in the Hitchin component, we have more information about the quotient $\rho(\Gamma)\backslash\Omega^2_\rho$. More explicitly, Guichard and Wienhard proved in [GW08] that the quotient of the domain of discontinuity by a Hitchin representation in $\mathrm{PSp}(4, \mathbb{R})$ gives rise to non-equivalent $(\mathrm{PSp}(4, \mathbb{R}), \mathbb{R}\mathbf{P}^3)$-structures, one being convex, the other not. Using the isomorphism $\mathrm{PSp}(4, \mathbb{R}) \cong \mathrm{SO}_0(2, 3)$, the homogeneous space $\mathbf{Pho}(\mathbb{R}^{2,3})$ of photons in $\mathbb{R}^{2,3}$ is identified with the space of lines in $(\mathbb{R}^4, \omega)$, where ω is a symplectic form on $\mathbb{R}^4$. In particular, a $(\mathrm{PSp}(4, \mathbb{R}), \mathbb{R}\mathbf{P}^3)$-structure is equivalent to a photon structure. We show the following

Proposition 5.6. *Given a Hitchin representation $\rho \in \mathrm{Hit}(\Gamma, \mathrm{SO}_0(2, 3))$, the photon structure on the degree $6g - 6$ circle bundle $\mathrm{SO}(U, V)$ constructed in Subsection 4.3 is equivalent to the non-convex projective structure described above, while the photon* *structure on the degree $-2g+2$ circle bundle $\mathrm{SO}(U,\overline{V})$ corresponds to the convex one.*

Proof. We will prove the result for the Fuchsian locus, which will give the result for the Hitchin component by continuity. Given $j:\Gamma\longrightarrow\mathrm{PSL}_2(\mathbb{R})$ a Fuchsian representation, let $\rho:=m_{irr}\circ j\in\mathrm{Hit}(\Gamma,\mathrm{PSp}(4,\mathbb{R}))$ be the image of j by the irreducible representation $m_{irr}:\mathrm{PSL}_2(\mathbb{R})\longrightarrow\mathrm{PSp}(4,\mathbb{R})$ corresponding to the action of $\mathrm{PSL}_2(\mathbb{R})$ on the space $\mathbb{R}_3[X,Y]$ os homogeneous polynomials of degree 3. Here, we identify $\mathbb{R}^2$ with $\mathbb{R}_1[X,Y]$ and $\mathbb{R}^4$ with $\mathbb{R}_3[X,Y]$. The convex connected component of Ω^2_ρ corresponds to those polynomials having a real root and two complex conjugate ones while the non-convex component corresponds to the set of polynomials having 3 distinct real roots.

The uniformization $u:\widetilde{\Sigma}\longrightarrow\mathbb{H}^2$ associated to j gives an equivariant identification $T^1\widetilde{\Sigma}\cong\partial_\infty{\mathbb{H}^2}^{(3)}$ where $\partial_\infty{\mathbb{H}^2}^{(3)}$ is the set of pairwise distinct triple $(x_-,x_t,x_+)\in(\partial_\infty\mathbb{H}^2)^3$ that are positively oriented. Indeed, given $(x,v)\in T^1\widetilde{\Sigma}$, there is a unique triple $(x_-,x_t,x_+)\in\partial_\infty{\mathbb{H}^2}^{(3)}$ such that the geodesic γ passing through $du_x(v)$ intersects $\partial_\infty\mathbb{H}^2$ in the future at x_+, in the past at x_- and the geodesic orthogonal to γ at x intersects the boundary at x_t with (x_-,x_t,x_+) positively oriented.

The developing map dev' corresponding to the non-convex projective structure is given by

$$dev':\begin{array}{ccc}T^1\widetilde{\Sigma} & \longrightarrow & \mathbb{R}\mathbf{P}^3\\ ([P_1],[P_2],[P_3]) & \longmapsto & [P_1P_2P_3]\end{array}.$$

Note here that dev' is invariant under the $\mathbb{Z}_3$-action on $T^1\widetilde{\Sigma}$ generated by $([P_1],[P_2],[P_3])\to([P_2],[P_3],[P_1])$. In particular, dev' descends to a ρ-equivariant injective map

$$\mathrm{dev}^*:T^1\widetilde{\Sigma}/\mathbb{Z}_3\longrightarrow\Omega^2_\rho.$$

The quotient of the image of dev^* by $\rho(\Gamma)$is thus a circle bundle of degree $(6g-6)$.

Note also that the developing map $\mathrm{dev}:T^1\widetilde{\Sigma}\longrightarrow\Omega^2_\rho$ corresponding to the convex foliated projective structure of Guichard–Wienhard is injective so the associated geometric structure is equivalent to the one on $\mathrm{SO}(U,\overline{V})$. □

5.3. **Equivalence of Einstein structures.** For a Hitchin representation $\rho:\Gamma\to\mathrm{SO}_0(2,3)$ there is a Guichard-Wienhard domain Ω^1_ρ in $\mathbf{Ein}^{1,2}$ by Proposition 5.1.

Guichard–Wienhard's theorem (Theorem 5.2) implies that the action of $\rho(\Gamma)$ on Ω^1_ρ is properly discontinuous and co-compact. Actually, one can be a bit more precise. Mimicking their construction of projective structures associated to Hitchin representations into $\mathrm{SL}(4,\mathbb{R})$ (see [GW08]), one can give[2] a ρ-equivariant parametrization of Ω^1_ρ by the set $\partial_\infty\Gamma^{(3)}$ of oriented triples of distinct points in $\partial_\infty\Gamma$. It follows that $\rho(\Gamma)\backslash\Omega^1_\rho$ is homeomorphic to $T^1\Sigma$. However, the circle bundle structure is not appearing in this construction.

Here, we prove that the conformally flat 3-manifold associated to ρ by Theorem 4.2 is isomorphic (as a conformally flat 3-manifold) to $\rho(\Gamma)\backslash\Omega^1_\rho$.

[2]This construction was done in some working notes that Guichard and Wienhard kindly shared with us.

Theorem 5.7. *Let $\rho : \Gamma \to \mathrm{SO}_0(2,3)$ be a Hitchin representation. Then the developing map dev_ρ constructed in Section 4.4 is a global homeomorphism from $T^1\widetilde{\Sigma}$ to Ω^1_ρ.*

The proof is less straightforward than that of Theorem 5.3. We first prove the following lemma, which settles the case when ρ is Fuchsian, and then argue by continuity, using the Ehresmann–Thurston principle.

Lemma 5.8. *Suppose that $\rho = m_{irr} \circ j$, where $j : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ is a Fuchsian representation and $m_{irr} : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{SO}_0(2,3)$ is the irreducible representation. The developing map dev_ρ constructed in Section 4.4 is a diffeomorphism onto Ω^1_ρ.*

Lemma 5.8 shows in particular that for $\rho_0 = m_{irr} \circ j$, the manifold $\rho_0(\Gamma)\backslash\Omega^1_{\rho_0}$ is homeomorphic to $T^1\Sigma$. Now, when ρ varies continuously, the topology of $\rho(\Gamma)\backslash\Omega^1_\rho$ does not vary, and its Einstein structure varies continuously by [GW12, Theorem 9.2]. Therefore, the developing map dev_ρ constructed in the proof of Theorem 4.2 and the identification of $T^1\Sigma$ with $\rho(\Gamma)\backslash\Omega^1_\rho$ given by Theorem 9.2 of [GW12] give two Einstein structures on $T^1\Sigma$ with the same holonomy ρ and depending continuously on ρ. Since the two Einstein structures coincide at $\rho_0 = m_{irr} \circ j$, they coincide on the whole connected component of ρ_0 according to the Ehresmann–Thurston principle. This concludes the proof of Theorem 5.7.

Proof of Lemma 5.8. The specificity of the Fuchsian case is that the developing map extends as a $\mathrm{PSL}(2,\mathbb{R})$-equivariant map from $T^1\mathbb{H}^2$ to $\mathbf{Ein}^{1,2}$.

Let us recall that the irreducible representation of $\mathrm{SL}(2,\mathbb{R})$ in dimension $n+1$ is given by the action of $\mathrm{SL}(2,\mathbb{R})$ on the space $\mathbb{R}_n[X,Y]$ of homogeneous polynomials of degree n in two variables X and Y. This action preserves the bilinear form Q_n given in the coordinate system

$$(X^n, X^{n-1}Y, \ldots, XY^{n-1}, Y^n)$$

by the matrix

$$\begin{pmatrix} & & & & a_{n,0} \\ & & & -a_{n,1} & \\ & & \iddots & & \\ & (-1)^{n-1}a_{n,n-1} & & & \\ (-1)^n a_{n,n} & & & & \end{pmatrix}$$

where $a_{n,k} = \frac{k!(n-k)!}{n!}$.

This bilinear form is anti-symmetric for n odd and symmetric of signature $(2k, 2k+1)$ for $n = 4k+1$. In particular, for $n = 2$, the quadratic form $-2Q_2$ is the discriminant of quadratic polynomials, and this representation gives the isomorphism $\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{SO}_0(2,1)$. The hyperbolic plane $\mathbb{H}^2$ thus identifies with the projectivisation of the set of quadratic polynomials with negative discriminant (that is, scalar products on $\mathbb{R}^2$) while $\partial_\infty\mathbb{H}^2$ identifies with the projectivisation of the set of quadratic polynomials with vanishing discriminant (that is, squares of linear forms).

Let $j : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ be a Fuchsian representation. We identify j with its composition with the isomorphism $\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{SO}_0(2,1)$. Now, $\mathbb{R}^{2,3}$ identifies with $(\mathbb{R}_4[X,Y], -Q_4)$, and the irreducible representation described above is the representation m_{irr}.

In this setting, the boundary map $\xi_0 : \partial_\infty \Gamma \to \mathbf{Ein}^{1,2}$ given by the Anosov property of ρ_0 is identified with the $\mathrm{PSL}(2,\mathbb{R})$-equivariant map

$$\begin{array}{rccc} \xi_0 : & \partial_\infty \mathbb{H}^2 & \to & \mathbf{Ein}^{1,2} \\ & [L^2] & \mapsto & [L^4] \ . \end{array}$$

(Here, $[L^2]$ denotes the projective class of the square of a linear form on $\mathbb{R}^2$.)

Moreover, given a point $[L^2]$ in $\partial_\infty \mathbb{H}^2$, the photon $\xi_1([L^2])$ is the tangent to ξ_0 at L. It is thus the projectivisation of the space of polynomials of the form $L^3 L'$, where L' is a linear form. We conclude that the domain $\Omega^1_{\rho_0}$ of Guichard and Wienhard is the complement in $\mathbf{Ein}^{1,2}$ of the set of polynomials having a triple root.

On the other side, the ρ_0-invariant maximal surface in $\mathbb{H}^{2,2}$ is the image of the $\mathrm{PSL}(2,\mathbb{R})$-equivariant map

$$\begin{array}{rccc} f : & \mathbb{H}^2 & \to & \mathbb{H}^{2,2} \\ & [P] & \mapsto & [P^2] \ , \end{array}$$

referred to as the *Veronese surface* in [Ish88]. (Here, $[P]$ denotes the projective class of a positive definite quadratic form on $\mathbb{R}^2$.)

Let P be a positive definite quadratic form on $\mathbb{R}^2$. The tangent space to this maximal surface at the point $f([P])$ is the projective space of polynomials of the form PQ, with $Q \in \mathbb{R}_2[X,Y]$. Since none of these polynomials has a triple root, the intersection of this tangent space with $\mathbf{Ein}^{1,2}$ is contained in the domain $\Omega^1_{\rho_0}$. By construction of the developing map dev_{ρ_0} it follows that dev_{ρ_0} takes values into $\Omega^1_{\rho_0}$.

Let $[P]$ and $[Q]$ be two distinct points in $\mathbb{H}^2$. Then the intersection between the tangent spaces to $f(\mathbb{H}^2)$ at $f([P])$ and $f([Q])$ is the point $[PQ]$, which never belongs to $\mathbf{Ein}^{1,2}$. Indeed, up to applying an element of $\mathrm{PSL}(2,\mathbb{R})$, one can assume that $[P] = [X^2+Y^2]$ and $[Q] = [aX^2 + bY^2]$. One easily compute that

$$Q_4(PQ) = \frac{1}{6}(a+b)^2 + 2ab \ ,$$

which never vanishes when a and b are of the same sign. By construction, it follows that dev_{ρ_0} is injective.

Let us finally prove that dev_{ρ_0} is surjective onto $\Omega^1_{\rho_0}$. Let P be a non-zero polynomial of degree 4 such $Q_4(P) = 0$. Suppose that $[P]$ is not in the image of dev_{ρ_0}. Then P is not divisible by a positive definite quadratic form and P thus splits as a product of 4 linear forms. If all these linear forms are co-linear, then $[P]$ belongs to the image of ξ_0 and thus not to $\Omega^1_{\rho_0}$. Otherwise, one can assume (up to applying an element of $\mathrm{PSL}(2,\mathbb{R})$) that P has the form

$$XY(aX+bY)(cX+dY) \ .$$

One then computes that

$$Q_4(P) = \frac{1}{6}\left((ad)^2 + (bc)^2 - adbc\right) \ .$$

Since the polynomial $A^2 + B^2 - AB$ is positive definite the fact that $Q_4(P)$ vanishes implies that both ad and bc vanish, from which we easily deduce that P is divisible by X^3 or Y^3. Therefore, P belongs to the complement of $\Omega^1_{\rho_0}$.

By contraposition, we deduce that, if $[P]$ belongs to $\Omega^1_{\rho_0}$, then P is divisible by a positive definite quadratic form. Therefore, the developing map dev_ρ is surjective onto $\Omega^1_{\rho_0}$. This concludes the proof of Lemma 5.8. □

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