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Dynamics of elastic rods in perfect friction contact

François Gay-Balmaz\(^1\) and Vakhtang Putkaradze\(^2\)

\(^1\) LMD - Ecole Normale Supérieure de Paris - CNRS, 75005, Paris; email: gaybalma@lmd.ens.fr
\(^2\) Department of Mathematical and Statistical Sciences
University of Alberta, Edmonton, AB T6G 2G1 Canada; email: putkarad@ualberta.ca

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Introduction Take two rubber strings, stretch them a bit and cross them so they remain in contact, as shown on Figure 1. As long as the deformations are not too large, the strings will roll at the contact without sliding. Of course, that simple experiment is dominated by the energy loss from the internal deformation of strings at the contact point; however, improving the quality of strings will allow them to oscillate for a reasonable time before the energy loss takes over. Interestingly, this simple and familiar experiment has deep mathematical and physical implications that go beyond a toy problem.

There are many objects that can be represented as long elastic rods, from a rubber hose to DNA molecules. Typically, if these objects are put in a confined space, or undergo other non-trivial dynamics, self-contact of these rods typically appears. The true dynamics will not allow the rod to pass through itself, and it must preserve the side of contact under dynamics. While it is generally accepted that something like DNA at contact will slide freely, the dynamics of other molecules like dendronized polymers (DP) may be different. These compound molecular structures are formed by assembling multiple dendrons that are each connected by its base to a long polymeric backbone [1]. A simplified, coarse-grained rod model of such polymers must take into account the rough surface formed by tree-like structures that is likely to generate perfect rolling contact. Another case when the rods will be rolling rather than sliding at contact can be realized for highly adhesive surfaces. A simple physical estimate demonstrates how essential that phenomenon is. Suppose two (macroscopic) elastic rods in contact have coefficient of dry friction of \(k\). For microscopic objects like DPs, the roughness of the rod’s surface will introduce an effective value of \(k\), although the exact value is not yet available in the literature. If the force due to contact \(\lambda\) (see below on how to compute that force) has a normal component to the surface \(\lambda_n\) and the tangential component \(\lambda_t\), then rolling contact occurs when \(|\lambda_t| \leq k|\lambda_n|\). Thus, whenever the contact force lies in the cone with the opening angle \(\arctan k\) with respect to the line connecting the centers, the contact will be rolling rather than sliding. For example, for a highly compacted configurations of a rod with many self-intersections, there will be a large fraction of contacts where the reaction force enforces rolling at contact; assuming that each contact generates a force that is uniformly randomly distributed within a hemisphere of possible forces enforcing contact, we conclude that the fraction of contacts experiencing rolling is \(\sim 0.5k/\sqrt{k^2 + 1}\). Even for the rather small value of \(k \sim 0.2\), approximately \(\sim 10\%\) of contacts will be rolling; larger values of \(k \sim 1\) gives that the rolling contacts will be encountered in \(\sim 30\%\) of cases.

Efficient numerical methods have been developed recently to deal with the self-contact forces of rods using short-range repulsive potential for statics [2] and dynamics [3–5]. Another avenue of studies of stationary states of elastic rods with self-contact [6–10] explicitly computes the contact forces from the existence of constraints. In our case, the rolling contact comes from friction, which does not admit any potential description. Thus, earlier works on rod dynamics based on nonlocal potential forces [11, 12] cannot be extended to the case of rolling contact. The extension of the latter approach to include the dynamics, and especially the rolling constraint is difficult.

FIG. 1: A sketch of two strings in contact. The line going through the centers of contact disks \(r_1\) and \(r_2\) passes through the contact point. Contact reaction force is denoted as \(\lambda\).
We also note that the true motion may be a combination of rolling and sliding friction, but in the absence of a consistent theory for rolling slippage we shall concentrate on the perfect rolling only. Our solution of the problem of rolling contact is similar in spirit of the second approach, and the contact force comes naturally from the constraint and not from short-range forces.

Setup of the problem As is well known [13], the problem of rolling motion is essentially non-holonomic and, in general, cannot be represented from the potential point of view. We shall note that it is possible to define potential approach to some non-holonomic systems [14]. However, these cases seem to be more of an exception rather than a rule. For the problem in hand, we proceed with the Lagrange-d’Alembert (LdA) principle which is the fundamental method for the treatment of non-holonomic systems [13, 15]. Similar approach has been recently used to describe the motion of an elastic rod rolling on a plane [16]. In order to utilize LdA, we recast the motion of the rods in variational setting through the Simo-Marsden-Krishnaprasad (SMK) rod theory [17], which allows a variational formulation of string dynamics [11]. The rod is parameterized by a coordinate \( s \) and the vector from the center of the cylinder \( i \) to contact, also in the fixed frame, is

\[
\mathbf{r}(s, t) = \mathbf{r}_i(s) - s_i(t) \mathbf{n},
\]

where \( \mathbf{n} \) is the unit vector normal to the cylinder at contact point \( i \). SMK theory formulates the Lagrangian associated with \( s_i(t) \) as

\[
\mathbf{L} = \mathbf{L}(\mathbf{r}, \mathbf{r}, \mathbf{r}, s_i)
\]

which is parameterized by a coordinate \( s \) and the vector from the center of the cylinder \( r_i(t) \) to contact point \( s_i(t) \), where \( i = 1, 2 \). We fix a reference frame and measure the distance \( s \) and \( s_i(t) \) marking the disks at contact. Since the angular velocity in the fixed frame is \( \dot{\mathbf{r}}_i(t) = \mathbf{r}_i(t) \), \( i = 1, 2 \), then the velocity of the material point associated with contact, also in the fixed frame, is

\[
\dot{r}_i + \frac{1}{2} \dot{\mathbf{r}}_i^2 = \dot{r}_i - 2 \dot{\mathbf{r}}_i + (r_2 - r_1) \tag{3}
\]

This condition can be reformulated in terms of invariant variables by multiplying (3) by \( \lambda_1^{-1} \):

\[
\gamma_1 + \frac{1}{2} \omega_2 \times \kappa_{12} = \xi_2 \gamma_2 - \frac{1}{2} (\xi_2 \omega_2) \times \kappa_{12} \tag{4}
\]

where \( \xi_{12} \) is the relative orientation, \( \kappa_{12} = \lambda_1^{-1}(r_2 - r_1) \), and other invariant variables are defined as in (4), with the index \( i \) meaning evaluation at \( s = s_i \). Note that due to the uniformity of the strand, and the assumption of circular cross-section, the Lagrangian does not depend explicitly on \( s_i \) and \( s_i \).

We also note that the true motion may be a combination of rolling and sliding friction, but in the absence of a consistent theory for rolling slippage we shall concentrate on the perfect rolling only. Our solution of the problem of rolling contact is similar in spirit of the second approach, and the contact force comes naturally from the constraint and not from short-range forces.
where the $2 \times 2$ matrix $A$ and 2-vector $v$ depends on the dynamical properties at contact, with $\det(A) \neq 0$ when the rods are not locally parallel at the contact. Equations (5,6) and the conditions (8,9) give the complete system of equations of elastic rods in rolling contact. These equations are different from earlier works as they take into account the forces caused by rolling conditions.

**Discrete strands in contact** It is also interesting to consider the application of this theory to discrete, chain-like elastic structures in contact. In that case, we need to clarify the physical meaning of the $\delta$-function at the contact position. Apart from its physical relevance, this consideration is also useful for consistent numerical discretization of (5,6). Here, care must be taken in deriving the equations of motion without breaking their variational structure [18]. Suppose that we have a string consisting of discrete set of points along the line, $s = s^k$, with $k$ being integer. If the orientation and position of a material frame at $s = s^k$ are given by an orientation matrix $\Lambda_k$ and a vector $r_k$, the invariant variables are $p_k = \Lambda_k^{-1} r_{k+1}$ and $q_k = \Lambda_k^{-1} (r_{k+1} - r_k)$. The purely elastic Lagrangian is $\ell = \ell(\omega_k, \gamma_k, p_k, q_k)$. One also needs to define a "smeared-out" version of (4):

$$\alpha_k (\gamma_k^r + \frac{1}{2} \beta_m \omega_k \times \kappa_{km})$$
$$- \alpha_k \beta_m (\xi_{km} \gamma_m - \frac{1}{2} (\xi_{km} \omega_m) \times \kappa_{km} = 0, \quad (10)$$

(summation over $k, m$). Here, we have defined the averaging coefficients: $\alpha_k := G(s_1 - s^k)$, $\beta_k := G(s_2 - s^k)$ arising from a "bump" function $G(s)$ that rapidly decays away from $s = 0$, and $\xi_{km} := \Lambda_k^{-1} \Lambda_m$, $\kappa_{km} := \Lambda_k^{-1} (r_m - r_k)$. Note that the positions of the contact are defined by the continuous variables $s_1$ and $s_2$. Here, the width of $G(s)$ corresponds to the size of discrete elements on the string or, for numerical discretization, to the distance along $s$ between the discrete points of the rod. The function $G(s)$ should satisfy three criteria: a) it is sufficiently smooth in order to avoid artificial accelerations of $s_2$; b) having a single maximum so there is no ambiguity about the position of contact; and c) decaying rapidly so as not to introduce any long-term interactions between the contact points on the rod. The physical meaning of (10) consists in spreading the point wise contact condition (4) to a few neighboring points surrounding the contact. Then, the $\text{LdA}$ principle gives a discrete analogue of (5,6):

$$\frac{D}{Dt} \partial \ell / \partial \gamma_k - p_{k-1}^{-1} \partial \ell / \partial q_{k-1} + \partial \ell / \partial q_k = \sum_m (\alpha_m \beta_m \xi_{km}^{-1} - \alpha_k) \lambda, \quad (11)$$

$$\frac{D}{Dt} \partial \ell / \partial \omega_k - p_{k-1}^{-1} \partial \ell / \partial p_{k-1} = \partial \ell / \partial \gamma_k \times \gamma_k + \partial \ell / \partial q_k \times q_k$$
$$- \sum_m \frac{1}{2} \left( \alpha_k \beta_m \kappa_{km} \times \lambda + \beta_m \alpha_m \xi_{km}^{-1} (\kappa_{mk} \times \lambda) \right). \quad (12)$$

For a Lagrangian that is quadratic in all variables, corresponding to linear elasticity, an explicit computation of $A$ in (11,12) is possible, but cumbersome. Equations (11,12) are augmented by conditions for the variables $s_1$ and $s_2$ similar to (9) obtained by differentiating a discrete version of (8). Note that one could have, in principle, guessed the equations of motion for continuum rods in contact (5,6) using the standard Kirchhoff model and physical intuition, but we do not see any way to derive equations (11,12) without using the methods of this paper.

**Linear strings in contact** One may wonder if the equations of motion we have derived allow to deduce analytical expressions for the propagation of the disturbances along the rods at contact, such as the dispersion relation. The answer to this question is, unfortunately, no: the contact condition makes the disturbances essentially nonlinear. Let us consider two strings in contact, and denote for shortness $a = (\gamma, \omega)^T$ and $A = (\Gamma, \Omega)^T$. For linear elastic materials $\partial^2 / \partial a = V a$ and $\partial^2 / \partial A = -Q A$, where $V$ and $Q$ are 6x6 matrices. The linearized compatibility conditions (7) allow to introduce a vector potential $\phi$ as $a = \partial_t \phi, A = \partial_x \phi$. Neglecting all nonlinear terms in the dynamic variables and assuming that the rod is naturally straight in its undeformed state, we can transform (5,6) into a vector wave equation [19]

$$V \frac{\partial^2 \phi}{\partial t^2} - Q \frac{\partial^2 \phi}{\partial s^2} = \left( \frac{1}{2} \kappa_{12} \times \right) \lambda s_1 + \xi_{12} \left( - \frac{1}{2} \kappa_{12} \times \right) \lambda s_2. \quad (13)$$

The condition (4) can be expressed in this vector form as

$$(\text{Id} - \frac{1}{2} \kappa_{12} \times)^T \phi_t(s_1, t) = \left( \xi_{12} \times \frac{1}{2} \kappa_{12} \times \xi_{12} \right) \partial_x \phi_t(s_2, t) \quad (14)$$

Thus, the evolution of small disturbances on the rod is governed by linear equations (13), linear rolling constraint (14) and nonlinear evolution equations for $s_{1,2}$ (9). We can illustrate the complexity of this problem on a pedagogical simplistic example of two straight rods in contact with only one mode being relevant in (13) for each rod. The one-dimensionality of disturbances is chosen just for the simplicity of computations. The chaotic behavior comes from the nonlinearity of coupling the forcing in equations (13) to the solution through the contact conditions (9), and is independent on the dimensionality of the system. Physically, such a mode can be realized for a rod with special elastic and inertia matrices $V$ and $Q$, e.g., for rods made out of composite material. No further simplifications of equations or analytical progress is possible. Thus, the answer to a deceptively simple question on how the disturbances propagate along the rod is surprisingly complex, and is due exclusively to the contact condition.

Let us denote by $u(x, t)$ and $v(y, t)$, the one-dimensional deflection for the first and second rod, respectively. In this case, the rolling constraint and the motion of the contact points are written simply as...
\[ u_i(s_1) = F v_i(s_2), \quad s_i = G_i u_i(s_1), \quad i = 1, 2, \] (15)

where \( F \) and \( G_i \) are constants depending on the material parameters and the base state of the rods. The equations of motion for \((u, v)\) in this reduced setting are:

\[ u_{tt} - c^2 u_{xx} + \lambda \delta s_1 = 0, \quad v_{tt} - c^2 v_{yy} - \lambda F \delta s_2 = 0, \] (16)

where \( \lambda \) enforces the first constraint of (15).

A discrete version of these equations can be derived, similarly to the full equation described above in (12). We shall emphasize that the complexity caused by the nonlinear contact conditions is the same in the full equation (13) and its one-dimensional counterpart (16). While the complex dynamics caused by the nonlinear rod equations have been well studied, as far as we are aware, there has been no work on the complex dynamics caused by the rolling contact condition. In the absence of constraints, the wave equations provide a simple harmonic oscillation of the string. However, when the constraint is present, the motion of the string is challenging and complex.

**Contact chaos** In the case when the boundary conditions for the strings are periodic, exact solution of equations can be found, which we omit here. In the more realistic fixed boundary conditions for \( u \) and \( v \), the behavior is highly complex. The motion, as one can show from equations (16), conserves energy; however, in reality, friction with air and, more importantly, rolling friction will lead to energy dissipation. It is nevertheless interesting to see the structure of that dynamics, with the nonlinearity obtained only from the contact condition. As we see from Figure 2, the system produces a complex spatio-temporal dynamics of both strings. It is also relevant to present another measure of complexity, computed from the dynamics of the base harmonic of \( u(x, t) \), call it \( \hat{u}_1(t) \), as a function of \( t \). If the string were vibrating in the air, the sound sufficiently far away from the string will primarily contain the contribution from the first harmonic. On Figure 2, right, we plot the time spectrum \( S(\omega) \) as a function of temporal frequency \( \omega \), obtained from the time signal of the first harmonic \( \hat{u}_1(t) \). Starting with \( u(x, 0) = \sin x \), a linear rod with \( 0 < x < 2\pi \) without rolling contact will generate a purely monochromatic signal; however, when the contact is present, there is a persistence of high overtones to the signal. A sound file in the supplementary materials gives the reader an impression about the quality of that signal. We have found out that the chaotic behavior caused by the contact condition persists for all initial conditions we have tried. The appearance contact-caused chaos is highly interesting and important for many physical applications, and yet it has not been discussed previously.

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**FIG. 2:** Left: the initial spatio-temporal evolution of one of the strings in contact, with the red line marking the point of contact. The color, dark blue to white, denotes the deviation of the string from its equilibrium position, from high positive to high negative. Right: spectrum \( S(\omega) \) vs \( \omega/\omega_0 \) of time signal produced by the lead harmonic (in \( x \)) of \( u(x, t) \), with \( \omega_0 = 2\pi c/l \) being the basic frequency of a string without contact.