Integrability properties and Limit Theorems for the exit time from a cone of planar Brownian motion

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Abstract

We obtain some integrability properties and some limit Theorems for the exit time from a cone of a planar Brownian motion, and we check that our computations are correct via Bougerol's identity.

Key words: Bougerol's identity, planar Brownian motion, skew-product representation, exit time from a cone.

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1 Introduction

We consider a standard planar Brownian motion[§] $(Z_t = X_t + iY_t, t \geq 0)$, starting from $x_0 + i0, x_0 > 0$, where $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ are two independent linear Brownian motions, starting respectively from x_0 and 0.
As is well known [ItMK65], since $x_0 \neq 0$, $(Z_t, t \geq 0)$ does not visit a.s. the point 0 but keeps winding around 0 infinitely often. In particular, the continuous winding process $\theta_t = \mathrm{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0$ is well defined. A scaling argument shows that we may assume $x_0 = 1$, without loss of generality, since, with obvious notation:

$$\left(Z_t^{(x_0)}, t \geq 0\right) \overset{(law)}{=} \left(x_0 Z^{(1)}_{(t/x_0^2)}, t \geq 0\right). \quad (1)$$

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§When we simply write: Brownian motion, we always mean real-valued Brownian motion, starting from 0. For 2-dimensional Brownian motion, we indicate planar or complex BM.

Thus, from now on, we shall take $x_0 = 1$.
Furthermore, there is the skew product representation:

$$\log|Z_t| + i\theta_t \equiv \int_0^t \frac{dZ_s}{Z_s} = (\beta_u + i\gamma_u)\Big|_{u=H_t=\int_0^t \frac{ds}{|Z_s|^2}}, \tag{2}$$

where $(\beta_u + i\gamma_u, u \geq 0)$ is another planar Brownian motion starting from $\log 1 + i0 = 0$. Thus, the Bessel clock H plays a key role in many aspects of the study of the winding number process $(\theta_t, t \geq 0)$ (see e.g. [Yor80]).
Rewriting (2) as:

$$\log|Z_t| = \beta_{H_t}; \quad \theta_t = \gamma_{H_t}, \tag{3}$$

we easily obtain that the two σ-fields $\sigma\{|Z_t|, t \geq 0\}$ and $\sigma\{\beta_u, u \geq 0\}$ are identical, whereas $(\gamma_u, u \geq 0)$ is independent from $(|Z_t|, t \geq 0)$.
We shall also use Bougerol's celebrated identity in law [Bou83, ADY97] and [Yor01] (p. 200), which may be written as:

$$\text{for fixed } t, \quad \sinh(\beta_t) \overset{(law)}{=} \hat{\beta}_{A_t(\beta)} \tag{4}$$

where $(\beta_u, u \geq 0)$ is 1-dimensional BM, $A_u(\beta) = \int_0^u ds \exp(2\beta_s)$ and $(\hat{\beta}_v, v \geq 0)$ is another BM, independent of $(\beta_u, u \geq 0)$. For the random times $T_c^{|\theta|} \equiv \inf\{t : |\theta_t| = c\}$, and $T_c^{|\gamma|} \equiv \inf\{t : |\gamma_t| = c\}$, $(c > 0)$ by using the skew-product representation (3) of planar Brownian motion [ReY99], we obtain:

$$T_c^{|\theta|} = A_{T_c^{|\gamma|}}(\beta) \equiv \int_0^{T_c^{|\gamma|}} ds \exp(2\beta_s) = H_u^{-1}\Big|_{u=T_c^{|\gamma|}}. \tag{5}$$

Moreover, it has been recently shown that, Bougerol's identity applied with the random time $T_c^{|\theta|}$ instead of t in (4) yields the following [Vak11]:

Proposition 1.1 *The distribution of $T_c^{|\theta|}$ is characterized by its Gauss-Laplace transform:*

$$E\left[\sqrt{\frac{2c^2}{\pi T_c^{|\theta|}}} \exp\left(-\frac{x}{2T_c^{|\theta|}}\right)\right] = \frac{1}{\sqrt{1+x}}\varphi_m(x), \tag{6}$$

for every $x \geq 0$, with $m = \frac{\pi}{2c}$, and:

$$\varphi_m(x) = \frac{2}{(G_+(x))^m + (G_-(x))^m}, \quad G_\pm(x) = \sqrt{1+x} \pm \sqrt{x}. \tag{7}$$

The remainder of this article is organized as follows: in Section 2 we study some integrability properties for the exit times from a cone; more precisely we obtain some new results concerning the negative moments of $T_c^{|\theta|}$ and of $T_c^{\theta} \equiv \inf\{t : \theta_t = c\}$. In Section 3 we state and prove some limit Theorems for these random times for $c \to 0$ and for $c \to \infty$ followed by several generalizations (for extensions of these works to more general planar processes, see e.g. [DoV12]). We use these results in order to obtain (see Remark 3.4) a new simple non-computational proof of Spitzer's celebrated asymptotic Theorem [Spi58], which states that:

$$\frac{2}{\log t}\,\theta_t \xrightarrow[t\to\infty]{(law)} C_1 \ , \tag{8}$$

with C_1 denoting a standard Cauchy variable (for other proofs, see also e.g. [Wil74, Dur82, MeY82, BeW94, Yor97, Vak11]). Finally, in Section 4 we use the Gauss-Laplace transform (6) which is equivalent to Bougerol's identity (4) in order to check our results.

2 Integrability Properties

Concerning the moments of $T_c^{|\theta|}$, we have the following (a more extended discussion is found in e.g. [MaY05]):

Theorem 2.1 *For every $c>0$, $T_c^{|\theta|}$ enjoys the following integrability properties:*

(i) for $p>0$, $E\left[\left(T_c^{|\theta|}\right)^p\right]<\infty$, if and only if $p<\frac{\pi}{4c}$.

(ii) for any $p<0$, $E\left[\left(T_c^{|\theta|}\right)^p\right]<\infty$.

Corollary 2.2 *For $0<c<d$, the random times $T^{\theta}_{-d,c}\equiv\inf\{t:\theta_t\notin(-d,c)\}$, $T_c^{|\theta|}$ and T_c^{θ} satisfy the inequality:*

$$T_c^{\theta}\geq T^{\theta}_{-d,c}\geq T_c^{|\theta|}. \tag{9}$$

Thus, their negative moments satisfy:

$$for\ p>0,\ \ E\left[\frac{1}{(T_c^{\theta})^p}\right]\leq E\left[\frac{1}{\left(T^{\theta}_{-d,c}\right)^p}\right]\leq E\left[\frac{1}{\left(T_c^{|\theta|}\right)^p}\right]<\infty. \tag{10}$$

Proofs of Theorem 2.1 and of Corollary 2.2
(i) The original proof is given by Spitzer [Spi58], followed later by many authors [Wil74, Bur77, MeY82, Dur82, Yor85]. See also [ReY99] Ex. 2.21/page 196.
(ii) In order to obtain this result, we might use the representation $T_c^{|\theta|}=A_{T_c^{|\gamma|}}$ together with a recurrence formula for the negative moments of A_t [Duf00], Theorem 4.2, p. 417 (in fact, Dufresne also considers $A_t^{(\mu)}=\int_0^t ds\exp(2\beta_s+2\mu s)$, but we only need to take $\mu=0$ for our purpose, and we note $A_t\equiv A_t^{(0)}$) [Vakth11]. However, we can also obtain this result by simply remarking that the RHS of the Gauss-Laplace transform (6) in Proposition 1.1 is an infinitely differentiable function in 0 (see also [VaY11]), thus:

$$E\left[\frac{1}{\left(T_c^{|\theta|}\right)^p}\right]<\infty,\quad \text{for every } p>0. \tag{11}$$

Now, Corollary 2.2 follows immediately from Theorem 2.1 (ii). ∎

3 Limit Theorems for $T_c^{|\theta|}$

3.1 Limit Theorems for $T_c^{|\theta|}$, as $c \to 0$ and $c \to \infty$

The skew-product representation of planar Brownian motion allows to prove the three following asymptotic results for $T_c^{|\theta|}$:

Proposition 3.1 *a) For $c \to 0$, we have:*

$$\frac{1}{c^2}\, T_c^{|\theta|} \underset{c\to 0}{\overset{(law)}{\longrightarrow}} T_1^{|\gamma|}. \tag{12}$$

b) For $c \to \infty$, we have:

$$\frac{1}{c}\, \log\left(T_c^{|\theta|}\right) \underset{c\to\infty}{\overset{(law)}{\longrightarrow}} 2|\beta|_{T_1^{|\gamma|}}. \tag{13}$$

c) For $\varepsilon \to 0$, we have:

$$\frac{1}{\varepsilon^2}\, \left(T_{c+\varepsilon}^{|\theta|} - T_c^{|\theta|}\right) \underset{\varepsilon\to 0}{\overset{(law)}{\longrightarrow}} \exp\left(2\beta_{T_c^{|\gamma|}}\right) T_1^{\gamma'}, \tag{14}$$

where γ' stands for a real Brownian motion, independent from γ, and $T_1^{\gamma'} = \inf\{t : \gamma'_t = 1\}$

Proof of Proposition 3.1:
We rely upon (5) for the three proofs. By using the scaling property of BM, we obtain:

$$T_c^{|\theta|} = A_{T_c^{|\gamma|}}(\beta) \overset{(law)}{=} A_u(\beta)\Big|_{u=c^2T_1^{|\gamma|}}$$

thus:

$$\frac{1}{c^2}T_c^{|\theta|} \overset{(law)}{=} \int_0^{T_1^{|\gamma|}} dv\;\exp\left(2c\beta_v\right)\;. \tag{15}$$

a) For $c \to 0$, the RHS of (15) converges to $T_1^{|\gamma|}$, thus we obtain part *a*) of the Proposition.

b) For $c \to \infty$, taking logarithms on both sides of (15) and dividing by c, on the LHS we obtain $\frac{1}{c}\;\log\left(T_c^{|\theta|}\right) - \frac{2}{c}\log c$ and on the RHS:

$$\frac{1}{c}\;\log\left(\int_0^{T_1^{|\gamma|}} dv\;\exp\left(2c\beta_v\right)\right) = \log\left(\int_0^{T_1^{|\gamma|}} dv\;\exp\left(2c\beta_v\right)\right)^{1/c},$$

which, from the classical Laplace argument: $\|f\|_p \overset{p\to\infty}{\longrightarrow} \|f\|_\infty$, converges for $c \to \infty$, towards:

$$2\sup_{v\le T_1^{|\gamma|}}\left(\beta_v\right) \overset{(law)}{=} 2|\beta|_{T_1^{|\gamma|}}.$$

This proves part $b)$ of the Proposition.

$c)$

$$T^{|\theta|}_{c+\varepsilon} - T^{|\theta|}_{c} = \int_{T^{|\gamma|}_{c}}^{T^{|\gamma|}_{c+\varepsilon}} du\ \exp(2\beta_u) = \int_0^{T^{|\gamma|}_{c+\varepsilon} - T^{|\gamma|}_{c}} dv\ \exp\left(2\beta_{T^{|\gamma|}_{c}}\right)\ \exp\left(2\left(\beta_{v+T^{|\gamma|}_{c}} - \beta_{T^{|\gamma|}_{c}}\right)\right)$$
$$= \exp\left(2\beta_{T^{|\gamma|}_{c}}\right) \int_0^{T^{|\gamma|}_{c+\varepsilon} - T^{|\gamma|}_{c}} dv\ \exp(2B_v), \qquad (16)$$

where $\left(B_s \equiv \beta_{s+T^{|\gamma|}_{c}} - \beta_{T^{|\gamma|}_{c}}, s \geq 0\right)$ is a BM independent of $T^{|\gamma|}_{c}$.
We study now $\tilde{T}^{|\gamma|}_{c,c+\varepsilon} \equiv T^{|\gamma|}_{c+\varepsilon} - T^{|\gamma|}_{c}$, the first hitting time of the level $c+\varepsilon$ from $|\gamma|$, starting from c. Thus, we define: $\rho_u \equiv |\gamma_u|$, starting also from c. Thus, $\rho_u = c + \delta_u + L_u$, where $(\delta_s, s \geq 0)$ is a BM and $(L_s, s \geq 0)$ is the local time of ρ at 0. Thus:

$$\tilde{T}^{|\gamma|}_{c,c+\varepsilon} = \inf\{u \geq 0 : \rho_u = c+\varepsilon\} \equiv \inf\{u \geq 0 : \delta_u + L_u = \varepsilon\}$$
$$\overset{u=\varepsilon^2 v}{=} \varepsilon^2 \inf\left\{v \geq 0 : \frac{1}{\varepsilon}\delta_{v\varepsilon^2} + \frac{1}{\varepsilon}L_{v\varepsilon^2} = 1\right\}. \qquad (17)$$

From Skorokhod's Lemma [ReY99]:

$$L_u = \sup_{y \leq u}((-c - \delta_y) \vee 0)$$

we deduce:

$$\frac{1}{\varepsilon}L_{v\varepsilon^2} = \sup_{y \leq v\varepsilon^2}((-c - \delta_y) \vee 0) \overset{y=\varepsilon^2\sigma}{=} \sup_{\sigma \leq v}\left(\left(-c - \varepsilon\frac{1}{\varepsilon}\delta_{\sigma\varepsilon^2}\right) \vee 0\right) = 0. \qquad (18)$$

Hence, with γ' denoting a new BM independent from γ, (16) writes:

$$T^{|\theta|}_{c+\varepsilon} - T^{|\theta|}_{c} = \exp\left(2\beta_{T^{|\gamma|}_{c}}\right) \int_0^{\varepsilon^2 T^{\gamma'}_1} dv\ \exp(2B_v). \qquad (19)$$

Thus, dividing both sides of (19) by ε^2 and making $\varepsilon \to 0$, we obtain part $c)$ of the Proposition. ∎

Remark 3.2 *The asymptotic result c) in Proposition 3.1 may also be obtained in a straightforward manner from (16) by analytic computations. Indeed, using the Laplace transform of the first hitting time of a fixed level by the absolute value of a linear Brownian motion* $E\left[e^{-\frac{\lambda^2}{2}T^{|\gamma|}_b}\right] = \frac{1}{\cosh(\lambda b)}$ *(see e.g. Proposition 3.7, p 71 in Revuz and Yor [ReY99]), we have that for* $0 < c < b$, *and* $\lambda \geq 0$*:*

$$E\left[e^{-\frac{\lambda^2}{2}\left(T^{|\gamma|}_b - T^{|\gamma|}_c\right)}\right] = \frac{\cosh(\lambda c)}{\cosh(\lambda b)} \qquad (20)$$

Using now $b = c + \varepsilon$, *for every* $\varepsilon > 0$, *the latter equals:*

$$\frac{\cosh(\frac{\lambda c}{\varepsilon})}{\cosh\left(\frac{\lambda}{\varepsilon}(c+\varepsilon)\right)} \overset{\varepsilon\to 0}{\longrightarrow} e^{-\lambda}.$$

The result follows now by remarking that $e^{-\lambda}$ *is the Laplace transform (for the argument* $\lambda^2/2$*) of the first hitting time of 1 by a linear Brownian motion* γ', *independent from* γ.

3.2 Generalizations

Obviously we can obtain several variants of Proposition 3.1, by studying $T^{\theta}_{-bc,ac}$, $0 < a, b \leq \infty$, for $c \to 0$ or $c \to \infty$, and a, b fixed. We define $T^{\gamma}_{-d,c} \equiv \inf\{t : \gamma_t \notin (-d, c)\}$ and we have:

- $\frac{1}{c^2}\, T^{\theta}_{-bc,ac} \overset{(law)}{\underset{c\to 0}{\longrightarrow}} T^{\gamma}_{-b,a}$.

- $\frac{1}{c}\, \log\left(T^{\theta}_{-bc,ac}\right) \overset{(law)}{\underset{c\to \infty}{\longrightarrow}} 2|\beta|_{T^{\gamma}_{-b,a}}$.

In particular, we can take $b = \infty$, hence:

Corollary 3.3 *a) For $c \to 0$, we have:*

$$\frac{1}{c^2}\, T^{\theta}_{ac} \overset{(law)}{\underset{c\to 0}{\longrightarrow}} T^{\gamma}_{a}. \tag{21}$$

b)For $c \to \infty$, we have:

$$\frac{1}{c}\, \log\left(T^{\theta}_{ac}\right) \overset{(law)}{\underset{c\to \infty}{\longrightarrow}} 2|\beta|_{T^{\gamma}_{a}} \overset{(law)}{=} 2|C_a|, \tag{22}$$

where $(C_a, a \geq 0)$ is a standard Cauchy process.

Remark 3.4 ***(Yet another proof of Spitzer's Theorem)***
Taking $a = 1$, from Corollary 3.3(b), we can obtain yet another proof of Spitzer's celebrated asymptotic Theorem stated in (8). Indeed, (22) can be equivalently stated as:

$$P\left(\log T^{\theta}_{c} < cx\right) \overset{(law)}{\underset{c\to \infty}{\longrightarrow}} P\left(2|C_1| < x\right). \tag{23}$$

Now, the LHS of (23) equals:

$$\begin{aligned} P\left(\log T^{\theta}_{c} < cx\right) &\equiv P\left(T^{\theta}_{c} < \exp(cx)\right) \equiv P\left(\sup_{u\leq \exp(cx)} \theta_u > c\right) \\ &= P\left(|\theta_{\exp(cx)}| > c\right) = P\left(|\theta_t| > \frac{\log t}{x}\right), \end{aligned} \tag{24}$$

with $t = \exp(cx)$. Thus, because $|C_1| \overset{(law)}{=} |C_1|^{-1}$, (23) now writes:

$$for\ every\ x > 0\ given,\ P\left(|\theta_t| > \frac{\log t}{x}\right) \overset{(law)}{\underset{t\to \infty}{\longrightarrow}} P\left(|C_1| > \frac{2}{x}\right), \tag{25}$$

which yields precisely Spitzer's Theorem (8).

3.3 Speed of convergence

We can easily improve upon Proposition 3.1 by studying the speed of convergence of the distribution of $\frac{1}{c^2}\, T_c^{|\theta|}$ towards that of $T_1^{|\gamma|}$, i.e.:

Proposition 3.5 *For any function $\varphi \in \mathcal{C}^2$, with compact support,*

$$\frac{1}{c^2}\left(E\left[\varphi\left(\frac{1}{c^2}T_c^{|\theta|}\right)\right]-E\left[\varphi\left(T_1^{|\gamma|}\right)\right]\right)\underset{c\to 0}{\longrightarrow}E\left[\varphi'\left(T_1^{|\gamma|}\right)\,\left(T_1^{|\gamma|}\right)^2+\frac{2}{3}\varphi''\left(T_1^{|\gamma|}\right)\,\left(T_1^{|\gamma|}\right)^3\right].\tag{26}$$

Proof of Proposition 3.5:
We develop $\exp(2c\beta_v)$, for $c \to 0$, up to the second order term, i.e.:

$$e^{2c\beta_v} = 1 + 2c\beta_v + 2c^2\beta_v^2 + \ldots.$$

More precisely, we develop up to the second order term, and we obtain:

$$\begin{aligned}
E\left[\varphi\left(\frac{1}{c^2}T_c^{|\theta|}\right)\right] &= E\left[\varphi\left(\int_0^{T_1^{|\gamma|}} dv\ \exp(2c\beta_v)\right)\right] \\
&= E\left[\varphi\left(T_1^{|\gamma|}\right)+\varphi'\left(T_1^{|\gamma|}\right)\,\int_0^{T_1^{|\gamma|}}\left(2c\beta_v+2c^2\beta_v^2\right)dv\right] \\
&\quad +\frac{1}{2}E\left[\varphi''\left(T_1^{|\gamma|}\right)\,4c^2\left(\int_0^{T_1^{|\gamma|}}\beta_v dv\right)^2\right]+c^2 o(c).
\end{aligned}$$

We then remark that $E\left[\int_0^t \beta_v dv\right]=0$, $E\left[\int_0^t \beta_v^2 dv\right]=t^2/2$ and $E\left[\left(\int_0^t \beta_v dv\right)^2\right]=t^3/3$, thus we obtain (26). ∎

4 Checks via Bougerol's identity

So far, we have not made use of Bougerol's identity (4), which helps us to characterize the distribution of $T_c^{|\theta|}$ [Vak11]. In this Subsection, we verify that writing the Gauss-Laplace transform in (6) as:

$$E\left[\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{\frac{1}{c^2}T_c^{|\theta|}}}\exp\left(-\frac{xc^2}{2T_c^{|\theta|}}\right)\right]=\frac{1}{\sqrt{1+xc^2}}\varphi_m(xc^2),\tag{27}$$

with $m=\pi/(2c)$, we find asymptotically for $c\to 0$ the Gauss-Laplace transform of $T_1^{|\gamma|}$. Indeed, from (27), for $c\to 0$, we obtain:

$$E\left[\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{T_1^{|\gamma|}}}\exp\left(-\frac{x}{2T_1^{|\gamma|}}\right)\right]=\lim_{c\to 0}\frac{2}{\left(\sqrt{1+xc^2}+\sqrt{xc^2}\right)^{\pi/2c}+\left(\sqrt{1+xc^2}-\sqrt{xc^2}\right)^{\pi/2c}}.\tag{28}$$

Let us now study:

$$\begin{aligned}\left(\sqrt{1+xc^2}+\sqrt{xc^2}\right)^{\pi/2c} &= \exp\left(\frac{\pi}{2c}\log\left[1+\left(\sqrt{1+xc^2}-1\right)+\sqrt{xc^2}\right]\right)\\ &\sim \exp\left(\frac{\pi}{2c}\left[c\sqrt{x}+\frac{xc^2}{2}\right]\right)\underset{c\to 0}{\longrightarrow}\exp\left(\frac{\pi\sqrt{x}}{2}\right).\end{aligned}$$

A similar calculation finally gives:

$$E\left[\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{T_1^{|\gamma|}}}\exp\left(-\frac{x}{2T_1^{|\gamma|}}\right)\right]=\frac{1}{\cosh\left(\frac{\pi}{2}\sqrt{x}\right)}\ , \tag{29}$$

a result which is in agreement with the law of $\beta_{T_1^{|\gamma|}}$, whose density is:

$$E\left[\frac{1}{\sqrt{2\pi T_1^{|\gamma|}}}\exp\left(-\frac{y^2}{2T_1^{|\gamma|}}\right)\right]=\frac{1}{2\cosh\left(\frac{\pi}{2}y\right)}\ . \tag{30}$$

Indeed, the law of $\beta_{T_c^{|\gamma|}}$ may be obtained from its characteristic function which is given by [ReY99], page 73:

$$E\left[\exp(i\lambda\beta_{T_c^{|\gamma|}})\right]=\frac{1}{\cosh(\lambda c)}\ .$$

It is well known that [Lev80, BiY87]:

$$\begin{aligned}E\left[\exp(i\lambda\beta_{T_c^{|\gamma|}})\right] &= \frac{1}{\cosh(\lambda c)}=\frac{1}{\cosh(\pi\lambda\frac{c}{\pi})}=\int_{-\infty}^{\infty}e^{i\left(\frac{\lambda c}{\pi}\right)y}\frac{1}{2\pi}\frac{1}{\cosh(\frac{y}{2})}\,dy\\ &\overset{x=\frac{cy}{\pi}}{=}\int_{-\infty}^{\infty}e^{i\lambda x}\frac{1}{2\pi}\frac{\frac{\pi}{c}}{\cosh(\frac{x\pi}{2c})}\,dx=\int_{-\infty}^{\infty}e^{i\lambda x}\frac{1}{2c}\frac{1}{\cosh(\frac{x\pi}{2c})}\,dx\ .\end{aligned} \tag{31}$$

So, the density $h_{-c,c}$ of $\beta_{T_c^{|\gamma|}}$ is:

$$h_{-c,c}(y)=\left(\frac{1}{2c}\right)\frac{1}{\cosh(\frac{y\pi}{2c})}=\left(\frac{1}{c}\right)\frac{1}{e^{\frac{y\pi}{2c}}+e^{-\frac{y\pi}{2c}}}\ ,$$

and for $c=1$, we obtain (30).
We recall from Remark 3.2 that (see also [PiY03], where further results concerning the infinitely divisible distributions generated by some Lévy processes associated with the hyperbolic functions cosh, sinh and tanh can also be found):

$$E\left[\exp\left(-\frac{\lambda^2}{2}T_c^{|\gamma|}\right)\right]=\frac{1}{\cosh(\lambda c)}\ , \tag{32}$$

thus, for $c=1$ and $\lambda=\frac{\pi}{2}\sqrt{x}$, (29) now writes:

$$E\left[\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{T_1^{|\gamma|}}}\exp\left(-\frac{x}{2T_1^{|\gamma|}}\right)\right]=E\left[\exp\left(-\frac{x\pi^2}{8}T_1^{|\gamma|}\right)\right], \tag{33}$$

a result which gives a probabilistic proof of the reciprocal relation in [BPY01] (using the notation of this article, Table 1, p.442):

$$f_{C_1}(x) = \left(\frac{2}{\pi x}\right)^{3/2} f_{C_1}\left(\frac{4}{\pi^2 x}\right).$$

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